THE TEACHING OF ARITHMETIC

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PREFACE

This book is a discussion of the aims and purposes of a course in arithmetic and of the methods of presenting each topic that should find a place in our elementary schools. Hence it is essentially a book for teachers, supervisors, and those preparing to teach, and thus meets the needs of a text-book for normal schools and teachers' reading circles.

The successful method of teaching any topic, however, depends upon a clear knowledge of the fundamental principles upon which the topic is based. Hence in the discussion of each topic, the fundamental principles involved are fully presented in connection with the method of the classroom presentation of the topic; in fact, they form a very essential element of the so-called method of teaching.

Several of the chapters are summaries of lectures given before teachers' associations and institutes and it has been the requests from such audiences for the printed lectures that has encouraged the author to bring them all together in this form in the hope that they will have a wider influence upon the teaching of arithmetic.

John C. Stone.

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CHAPTER I

GENERAL PRINCIPLES AND SUGGESTIONS

Economy and Efficiency

The most important problems in the teaching of arithmetic are those of economy and efficiency. How to put the child in possession of the essential facts of the subject, how to develop habits that lead to the economical and efficient use of these facts in the real situations that arise in everyday life, and how to accomplish this with a minimum of waste are the problems that confront every thoughtful teacher.

Number Work Begun Too Early

Pestalozzi, in Europe, and Colburn, in our own country, are recognized as the leaders of the movement to bring instruction in arithmetic down into the very lowest grades and within the grasp of children. As an excuse for this, Colburn says, in the preface of his *First Lessons*: "As soon as children have the idea of more or less, and the names of a few of the first numbers, they are able to make small calculations. And this we see them do every day about their playthings, and about the little affairs

which they are called upon to attend to. * * * To succeed in this (the science of numbers), however, it is necessary rather to furnish occasions for them to exercise their own skill in performing examples, rather than to give them rules. * * * By following this mode, and making the examples gradually increase in difficulty, experience proves that at an early age children may be taught a great variety of most useful combinations of numbers." All observers of children will agree with Colburn that they are able to make simple calculations about their playthings and about their own little affairs. The work, however, should stop here until wider experiences and further needs demand greater knowledge of numbers. In the past we have begun number work entirely too early and have forced adult applications of number before the needs, interests, and experiences of the child were ready for them. We have tried to force facts and processes not needed in the natural activities of childhood, either in or out of school. Thus we have violated every principle of economy in teaching in not finding the child's needs and interests and teaching the right thing at the right time.

The Child's Need of Number

The child has certain needs of number during the first year of his school life that should be met. But, in general, these needs are confined to counting by ones, tens, and fives to one hundred; to reading numbers to 100 in order to find a page in his book, or perhaps to 1000 in order to find a house number if he lives in a city; to the few

simple combinations that he may need in his own little affairs; to the meaning of halves, thirds, and fourths of single objects; to the Roman numerals to XII, in order to tell time; and to those common units of measure that are needed in his home or school life. He has no natural need of any of the written processes yet, and, in fact, he makes figures with such conscious effort and knows automatically so few facts that to give written work before the third grade is to establish habits of computation that must be overcome later.

Educational Principles Involved

The economic mastery of the subject of arithmetic demands that the teacher observe the principles that knowledge to be real must be founded upon the actual experiences of the individual learner; that knowledge to be retained must be given an opportunity for use; and that a necessary condition for learning is that the process be self-actuated through motive or interest. Hence, the subject must be made to minister to the child's needs and kept within his actual experiences. It is when we select material from the world of adults, dealing with matters with which children have had no experiences, and in which they have no interest, that we are asking the impossible of children.

More Drill Work Needed

There are certain essential facts and processes that must be made automatic. It is not sufficient that a child knows that 7 and 8 are 15, but he must recall automatically

the fact when seeing the figures or hearing the numbers called, without consciously recalling the meaning of the figures or the "how many" denoted by the numbers, just as he sees letters and recalls words without recalling the sounds represented by the letters. The fundamental facts are clarified or rationalized by some concrete object or by a problem of conditions very familiar to the child. But when the meaning of the facts, processes, and terms used has thus been made clear, next in order is drill with abstract numbers until these facts or processes become automatic. Sight drills must be used before mental drills so as to furnish a mental image for the mental work. Suppose there are placed upon the blackboard the figures $\begin{array}{r}7\\ \underline{8}\\ 15\end{array}$. The child images this combination and when later he sees $\begin{array}{r}7\\ \underline{8}\end{array}$, he recalls 15. Then when he hears "seven and eight," he has a mental picture of these forms and can recall the sum. Experiment shows that the average person cannot grasp a group of objects larger than four or five without analysis; that is, without breaking them up into groups and adding the groups. Then the idea of a number as large as $8 + 7 = 15$ is not obtained through objects; that is, the fact is not made automatic through seeing objects, but through seeing the figures or from hearing "eight and seven are fifteen" until a mental image is formed. The objects, then, are to make clear the *meaning* of a fact or process; but they do not help the memory to retain such a fact.

HABITUATION *VS.* RATIONALIZATION

The subject of arithmetic is taught very largely to-day by inductive methods; that is, the child is led through simple concrete illustrations and objective presentations to discover his own facts, rules, and definitions; but there is danger that this phase of the work may receive undue emphasis. It frequently happens that a teacher is able to develop a subject very clearly and interestingly, and yet her work may lack effectiveness, owing to her neglect of drill work necessary to fix the facts and to give skill in computation.

There should be a sharp discrimination between those facts, the rationalization of which is of vital importance, and those in which it makes but little, if any, real difference as far as the efficient use of the facts is concerned. Thus it is of vital importance that a pupil know the meaning of addition and of such expressions as "five and four are nine," etc.; but it makes little, if any, difference whether or not he knows *why* we "carry." The important thing is that he has the proper *habit* of carrying. Likewise, in "long multiplication" he must have the habit of putting the partial products where they belong; but it makes little, if any, difference, when first presented, whether he knows *why* they are so placed.

When rationalization aids the memory in retaining a fact, or when it renders the use of it more safe when applied to problems, then rationalization should receive proper emphasis; otherwise, it is of minor importance. But in such cases as "carrying," "borrowing," and the

correct placing of partial products, the proper habit is the important thing and rationalization a very unimportant thing when the subject is first presented. On the other hand, the rationalization of a rule in fractions renders the use of it more safe and aids the memory in retaining the "how." It is not uncommon to see grammar school and high school pupils add $\frac{3}{5}$ and $\frac{2}{7}$ and get $\frac{5}{12}$, adding numerators and denominators. Proper rationalization should have prevented such an error.

Likewise, all rules for mensuration are more easily remembered if rationalized through an objective presentation.

Standards of Speed and Accuracy

In the past we have had no standards of skill in computation. Each teacher has had her own standards. Pupils have been marked and passed by these standards. One teacher has emphasized the "how" to solve a problem, regardless of the time required to compute the result or the accuracy of it. Others have emphasized merely the power to compute. Recent tests show startling variations in the work of pupils from schools reputed to be of the very highest standing. Such tests as the "Courtis tests," when given to enough pupils, may, by taking a general average, show the average of what schools are attaining by present methods; but they in no way show what may be expected of pupils by properly devised drills. From government statistics we may find the *average* production of wheat or corn for a state or the nation; but that does not show what should be

expected from improved methods of cultivation or the most economical yield. If careful records are kept of the progress made by some such methods of teaching and system of drills as are suggested in this book, we may ultimately know what should be expected of a pupil in accuracy and rapidity in the various grades. If some other method shows different results, we may thus judge the efficiency of the method used and thus have a means of judging the real value of various methods. At present, however, a method is considered good or bad by a teacher as it meets or fails to meet the particular whim of that individual teacher.

CHAPTER II

THE AIM OF A COURSE IN ARITHMETIC

Importance of an Aim

Our methods of teaching, and the nature of the subject matter that we teach, depend very largely upon our belief in what the subject may do for the learner; for, if we believe that our pupils should obtain certain results from our teaching, it naturally follows that we are going to shape our instruction and select our material in the way that seems to us best fitted to bring about those results.

For example, if a teacher considers that the only purpose of arithmetic is to train pupils to compute skillfully, she will reduce her instruction to drills that will bring about efficiency in computation. If problems are given at all, they will be given merely in order to furnish an occasion for computation, and their nature will not be given serious attention. On the other hand, if she considers that the purpose is "mental discipline," then she will pay less attention to computation and to the mechanical side of the subject and spend a great part of the time in developing and rationalizing the processes and in selecting problems having just the proper complexity to task the child's powers to discover what to do. The

problems will not be selected for any real purpose that they can serve in the child's life, but they will be selected to furnish a sort of mental gymnastic exercise. Or, she may consider that the whole purpose of arithmetic is to furnish a social insight into current business or industrial practices. In this case, she will devote most of her time to *real* problems that come up in life, regardless of the mathematics employed, and spend much time in discussing the social phases of situations from which the problems were made, disregarding drills and the accuracy of the solutions. And thus we might go on with one hypothetical case after another to show that a discussion of the aims of a course in arithmetic becomes of fundamental importance in discussing methods of teaching.

Traditional Values of the Study of Arithmetic

The values of arithmetic have varied with the development of the race and have depended upon the civilization of the people using the subject. The knowledge of numbers must have been coeval with the human race. With the notion of distinctness — that *this* is separate from *that* — the notion of number must have had its origin. One can scarcely carry on the simplest conversation or make any communication with another without a use of some knowledge of number. The savage on returning from the hunt no doubt communicated to his family or the tribe the amount of game he had seen or killed, possibly by the use of his fingers or hands or by pebbles or shells. To him, arithmetic was but a matter of convenience. But when he began to barter skins or

game for corn or trinkets and began to acquire property, there grew up a necessity for a knowledge of the relation of numbers — a comparison of the number of one group with that of another, or a comparison of their values. His arithmetic now became a necessity — it took on a utilitarian value.

As the race advanced in civilization and schools for the training of youths sprang up, arithmetic took on a new meaning. It was given a disciplinary value as well as a utilitarian value. But, just as our real need of number has grown up slowly and naturally and has thus gradually changed the utilitarian values of arithmetic, so has our knowledge of the laws of mental growth changed our thoughts about the disciplinary value of the subject.

Arithmetic was once taught to make one quick-witted and enable him to see a point quickly and thus aid him in argumentation. This concept of the value of arithmetic led naturally to catch-questions and puzzles.

Later there grew up under the teaching of the so-called "faculty" psychology the doctrine of "formal discipline." This was a belief in the transfer value of abilities developed by one study to all other lines of thought, whether related or not. This led to studying a subject in order to train the reason, or the memory, or the imagination, etc. This doctrine, however, did not result in any great change in the nature of the problems formerly given, but it merely gave a new reason for retaining the type of problems already in use. But it did make a great change in the methods of teaching. There was a change from the old dogmatic rule method

to the demonstrated rule method, and later to the heuristic or development method of presenting the facts and processes. While a belief in the doctrine of "formal discipline" undoubtedly resulted in improved methods of teaching over those methods that preceded it, it nevertheless placed undue emphasis upon the rationalization of the facts and processes, particularly in the lower grades, and it caused many obsolete topics and processes to be retained in the schools long after they could be justified upon practical or social grounds.

Doctrine of Formal Discipline Denied

A few years ago psychologists began to doubt the doctrine of formal discipline and to deny the transfer of abilities gained in one field of work to unrelated fields. By many teachers this new attitude of psychologists was entirely misunderstood. Much careless teaching and indifference toward the subject of arithmetic resulted. A teacher recently said to the writer that her class in arithmetic was very poor, and then added, "But if they are poor in anything, I suppose that it had better be in arithmetic than in anything else." Then she asked, "Hasn't arithmetic been taught in the past for nothing but discipline, and now hasn't it been found that there is no such thing?" It is just such an attitude as this, resulting from a misunderstanding of recent educational discussion, that has seriously affected the teaching of arithmetic.

Our modern psychology views our mental life as made up of specialized abilities and not of separate faculties, each trained by some special kind of study. The result

is that we now train for habits of accuracy, of attention, of analysis, of comparison, of judgment, etc., each in its own field. It follows, then, that each topic taught must have some bearing upon the life of the individual as a sort of socializing factor in order to justify its place in the curriculum.

The Practical Values of Arithmetic

As would naturally appear from the discussion of the last topic, the emphasis in arithmetic is now placed upon the practical values of the subject. That does not mean that all the older aims, as discipline, pleasure, culture, and preparatory values, are now wholly ignored; but it does mean that the emphasis has shifted from these to the practical in the broad sense of that term. But in so doing, whatever claims of recognition any of the older values have may be taken care of when teaching the subject from the practical standpoint.

But, like the term "formal discipline," the term "practical values" is very much misunderstood and misused. Some would view the practical side as relating to nothing but the ability to do the computing necessary in one's vocation. Others think that it relates to fitting one for some specialized vocation, usually a clerkship in some commercial pursuit. And there are still others that consider the practical as referring to any *real* problem met in any of the world's varied activities. Many of the problems that have found their way into the schools in recent years under the guise of "practical" are just as impractical and far more ridiculous than some of the

problems of the past which were given for recreation and mental gymnastics.

But, by a more careful analysis of the practical values, from the standpoint of the general user of arithmetic in society rather than the needs of the man in some specialized vocation, it is easily seen that the following things are practical to the average person in any walk of life:

1. Efficiency in computation.

2. A social insight into business and industrial practices that will enable one to interpret references to such practices met in general reading or in social and business intercourse.

3. Power to express and to interpret the numerical expressions of the quantitative relations that come within our experiences.

4. The habit of seeing such relations, particularly those that are vital to our welfare.

While to a degree these four abilities are of importance to all, their values vary with the different users of the subject.

Efficiency in Computation

Computation is not an end in itself, but a means to an end. Just as ability to form the letters of the alphabet is necessary if we are to express our thoughts in writing, but useless unless we have thoughts to express, so ability to compute is of no value unless we know what process to apply to a problem, for there is no relation between ability to compute and ability to reason out what process to apply to a given problem. This is sometimes misunderstood, and drill in pure computation is overem-

phasized in the thought that we are thus teaching arithmetic.

Since mechanical ability, like computation, is more easily measured than other results of teaching, this phase is likely to be overemphasized, particularly in those schools where "standard tests" are too closely relied upon in measuring the work of the school. However, there is need for accuracy and reasonable speed in using the four fundamental processes in whole numbers, fractions, and decimals; skill in mental calculations; and ability to see approximate results without a pencil.

The absolute control of the decimal point must be developed. The lack of such a control is the cause of numerous errors. Recently I saw a class attempt to find the cost of 2.9 cubic feet of gas which they had used in an experiment, knowing that the price was \$.90 per 1000 cubic feet. The class was divided into about three equal groups giving 26.1¢, 2.61¢, and .261¢ respectively as the answer.

Efficiency in computation, however, requires more than a mere automatic control of number facts and processes. Through a knowledge of the fundamental principles involved, pupils of the upper grammar grades should have the power to see relations that will save figures and even whole processes. Thus, if one is to find what per cent a gain of \$2,346,275 is of the total sales of \$86,724,342, he should be able to see that not half of the figures given need to be used in the calculation. Or, if he is to find the interest at 6 per cent by multiplying the principal by the number of days, pointing off three

more decimal places, and dividing by six — a very common method — he should see that, if the number of days is a multiple of six, the division may be saved and the result found by multiplying by one sixth of the number of days, and pointing off three more places.

Social Insight into Business and Industrial Practices

The proper study of arithmetic should develop an ability to interpret and comprehend the quantitative problems that arise in the world's varied activities. Whether one is to be actively engaged in a commercial or industrial vocation or not, in his daily reading and in intercourse with others he will meet references to many of the problems of commercial and industrial activities, and he must be able to interpret them if he is to take an important place in society. So the problems of arithmetic are now taking on a much more *real* nature than those of the past did. In a discussion of borrowing, loaning, and investing money, of taxes, insurance, etc., the indirect problems of the past are fast giving way to those problems that one actually needs to ask himself in any of these phases of social life.

For some students, the power to interpret and comprehend such problems has no further practical value than the enrichment of their understanding of the life about them. But to a vast majority of the students, there is a more direct vocational value; for, in our present commercial and industrial competition and struggle, other things being equal, the one having the keenest sense of

number relations has by far the greatest chance of success.

Power to Express and Interpret Quantitative Relations

The relations between numbers representing magnitudes are either differences or ratios. Of the two means of expressing relations, the ratio, expressed as a fraction or a per cent, gives the most definite or vital idea or picture. For example, to say that in 1915 the world's production of cotton was 14,126,500 bales, of which the United States produced all but 2,934,700 bales, creates no mental picture and means but little to us. But, to give a more definite picture, we say that the United States produced about four fifths of the world's crop of cotton in 1915.

The task of developing power to see and to interpret quantitative relations is a most difficult one. It can only be done by making a constant application to the quantitative aspects of life. While many feel that we have lost a means of developing such power by discarding the kind of problems found in the older types of "mental" and "intellectual" arithmetic, it seems reasonable that a closer correlation of arithmetic with the actual quantitative situations that arise in other school activities and in the everyday life of the child will come more nearly developing an ability that is of real use in interpreting the quantitative phases of life and in doing the world's work than can possibly come through artificial problems made up merely for mental gymnastics.

Elimination of Topics

A discussion of the values of a course in arithmetic naturally leads to a discussion of what is worth while in the course and what may be eliminated as not contributing to any of the purposes for which arithmetic is taught. While there can be no definite statement made that will meet every case, since much depends upon the individual user of arithmetic, but few, if any, in the ordinary walks of life will find much use for the following:

1. *The greatest common divisor.* — The only application of the greatest common divisor that is found in elementary arithmetic is its use in reducing a fraction to its lowest terms. But, if the greatest common divisor is to be found by the factoring method, it will be more economical to divide out the common factors as they are found than to take their product and then divide both terms by this product.

2. *Addition and subtraction of fractions with large or unusual denominators.* — On account of the nearly universal use of decimals, the addition and subtraction of fractions, except the very simplest ones, rarely ever occurs in practical work. Even those fractions and mixed numbers that are added and subtracted usually arise from expressing some number as 5 bu. 3 pk., 3 lb. 7 oz., etc., in terms of a single unit as $5\frac{3}{4}$ bu., $3\frac{7}{16}$ lb., etc.

The fractions with larger terms, used to express a ratio, are never added. Such ratios, however, are usually expressed decimally.

3. *The least common multiple.* — In elementary arithmetic, the only application of the least common multiple

of two or more numbers is its use in reducing fractions to common denominators. So, if only the fractions found in the practical problems of everyday life are given, the least common multiple is not needed, for the common denominators can be found more quickly by inspection.

4. *The more complex forms of complex fractions.* — It has become common among writers in recent years to recommend unreservedly the elimination of all complex fractions. However, a careful examination into the nature and use of the complex fraction causes a modification of such a recommendation. The relations wanted among fractions or mixed numbers are quite as common as those among whole numbers; and a complex fraction is but a ratio or an expressed division between two fractions or mixed numbers or a whole number and a fraction or a mixed number. Hence, the following ratios, expressed as fractions, are as apt to occur as those between whole numbers, namely:

$$\frac{\frac{1}{2}}{\frac{3}{5}},\quad \frac{3\frac{1}{2}}{8\frac{1}{4}},\quad \frac{\frac{3}{4}}{2\frac{1}{2}},\quad \frac{4}{3\frac{1}{2}},\quad \frac{2\frac{1}{3}}{5},\quad \frac{5\frac{1}{2}}{\frac{3}{4}}.$$

It may be urged that, since these are but special forms of expressing a division, why call them fractions, and why not simplify them as in ordinary division of fractions. But perhaps by writing them in this form and applying the principle that *Multiplying both terms of a fraction by the same number does not alter the value of the fraction* is the most economical way of simplifying such ratios. Thus, at sight, the above "complex fractions" become:

$$\tfrac{5}{6},\quad \tfrac{14}{33},\quad \tfrac{3}{10},\quad \tfrac{8}{7},\quad \tfrac{7}{15},\quad \text{and}\quad \tfrac{22}{3}.$$

5. *Obsolete tables and those used in specialized vocations.* — Under these tables would be included apothecaries' weight, troy weight, surveyors' measure, folding of paper, foreign money, and also the gill, furlong, rood, dram, quarter in avoirdupois weight, etc.

6. *Impractical reduction in denominate numbers.* — Reductions of more than one or two steps rarely, if ever, occur. One might have occasion to reduce feet to inches or inches to feet, but there is not apt to arise a need of changing miles to inches or inches to miles.

7. *Addition, subtraction, multiplication, and division of compound denominate numbers.* — In practical problems, a measure is expressed as a whole number, a mixed number, or a decimal in *one unit* instead of as a compound denominate number, before it is added, subtracted, multiplied, or divided. Hence, these topics need not receive any attention in the schoolroom.

8. *The present type of inverse problems in fractions and percentage.* — Most of the inverse or indirect problems given in fractions or percentage could not meet a real need of any one. They are a sort of "hide-and-go-seek" kind given for "exercises in analysis." Thus, in the problem: "If Mr. A sold a suit for $24, thereby making 20 per cent of the cost, find the cost," the answer cannot meet a real need of Mr. A, for he must have known the cost before he could have furnished the data of the problem.

It may occur, however, that Mr. A has a suit of which he knows the cost and wishes to know a selling price that will yield a certain per cent of itself. This would involve

the indirect type of problem. Usually, however, competition determines the price.

There are perhaps sufficient reasons for bringing up the inverse or indirect type of problems in a high school course; but there is much more profitable work for the grammar school pupil, at least before the eighth grade, than even the real applications of the indirect problem.

9. *The various short methods of finding interest.* — The one who has to compute interest often, as a clerk in a bank, makes use of a book of tables. One who computes interest but infrequently is apt to forget any of the various short methods; but, if he knows the meaning of interest, he cannot forget the general method, — that is, that the principal multiplied by the rate, and that product multiplied by the time expressed as a fraction of a year, gives the interest.

10. *All inverse problems of interest.* — The inverse or indirect problems of interest do not occur in practical life, and do not make more clear the meaning of interest, hence they can be justified only "for analysis." But that justification only is not sufficient, for there are plenty of problems that furnish sufficient drill in analysis while also serving a more important purpose.

11. *Partial payments.* — The subject is not of sufficient importance to justify it from any of the purposes of arithmetic.

12. *Annual interest.* — This cannot be justified from the standpoint either of practical uses or of social insight.

13. *Undue emphasis upon the discounting of interest-bearing notes.* — Beyond knowing the meaning of Bank

Discount — that is, that when a bank collects the interest in advance, this interest is called Bank Discount — drill in finding discount is not worth while.

14. *True discount.* — As a business custom, this is obsolete. Yet, it is given in many of the textbooks still in use.

15. *Partnership.* — This does not lead to an insight into business practices, for it is obsolete except in the smaller businesses; and the problems usually given lead to an erroneous notion of the division of profits even in a partnership business.

16. *Proportion as a general method of solving problems.* — Other methods of analysis and solution are not only shorter but develop greater power to see and express quantitative relations.

17. *Foreign and domestic exchange.* — Except the knowledge of how indebtedness may be canceled by check or draft instead of by an actual transfer of money, the subject is of no real practical value to the general student of arithmetic.

18. *The measurement of uncommon areas and volumes.* — The measurement of such areas and volumes as those of trapezoids, frustums, wedges, spheres, etc., are so uncommon in practical everyday life that the method of finding them is soon forgotten.

19. *Square root and the Pythagorean theorem.* — One of these topics is usually given as an excuse for giving the other. While they have but little practical value to the general user of arithmetic, reference to them is yet so common in general reading that a mere acquaintance with what they mean may justify a brief discussion of

them. It is better, however, to delay these topics until the high school course.

20. *The metric system.* — Unless the metric system is going to be needed in science, its teaching cannot be justified. It is sometimes urged as a means of bringing about a general use of the system, but it is difficult to show its advantages in an appealing way to grammar school pupils. The teaching of the subject should be delayed, then, until it is met in science.

The Essential Work in Arithmetic

There are those who seem inclined to think that the elimination of these twenty topics means a greatly reduced time to be given to arithmetic, but this is not the case. On every hand, one hears from the business world the complaint that grammar school, high school, and college students know but little arithmetic; that is, that the arithmetic of the schools does not "function" in actual practice. On every hand teachers are either asking for or suggesting a remedy. No panacea has yet been given. But it seems that the following suggestions may help bring about better results:

1. *Develop better habits of accuracy.* — From the very beginning of the work, pupils should be given methods of checking their work, and all work should be carefully checked; also all work handed in should be 100 per cent accurate in computation. The same standard of speed cannot be expected of all pupils, but all can work at a piece of computation until they are assured of its accuracy, just as an accountant does.

2. *Develop reasonable speed.* — There is some danger that too great an effort is made to get a fixed speed. In fact, too conscious haste upon the part of the pupil may lead to injurious results. The purpose of speed is largely to save time, so we must ask if the time saved is worth the effort to attain it.

3. *Encourage short methods.* — Pupils should understand fundamental principles that will lead to a saving of figures and processes. This is a slow and gradual growth that comes about with greater maturity and a greater insight into fundamental principles and numerical relations.

4. *Emphasize oral work.* — In life, one uses much more oral than written arithmetic. Pupils should learn to work without a pencil and to give either *exact* answers or *very close approximations* to all types of problems that arise in everyday life.

5. *Habituate rather than rationalize those facts and processes of frequent recurrence.* — When a fact or process is used without variation and arises very frequently in the life of the average student of arithmetic, drill should continue until the recalling of the fact or process becomes a habit. But the reason for the fact or process may well be delayed or omitted entirely. Thus, the primary facts and the fundamental processes with whole numbers must be made automatic, but to discuss *why* we "carry" or *why* we "borrow" is unnecessary when the work is first taken up.

6. *Avoid stereotyped solutions.* — The solution of a problem should never be an act of memory but always

a process of reasoning. Otherwise, the pupil is not gaining power to see relations but merely developing skill in juggling figures.

7. *Use the quantitative situations that arise in all other activities in or out of school.* — Every process as learned should find an immediate use in answering some real inquiry of the pupil about the quantitative relations that are vital to his interests.

CHAPTER III

COUNTING

Rote and Rational Counting

The idea of number is coeval with the human race. With the ability to distinguish differentia — that *this* is not *that* — originated the concept of number. But the first concept is the "how many" idea of number, not the "how much." The basis for the number facts is counting. Hence, the first step in the teaching of numbers is counting, first by rote, then rationally. In other words, the child first gets the order or sequence of the numbers and is able to say, "one, two, three, four, five, six," etc., without being able to recognize any of these numbers of things. Then from objects he learns the meaning of these names; that is, the counting becomes rational.

When pupils have trouble in getting the order of the numbers fixed, or when they have developed a wrong order, as "one, two, three, five, eight, nine," etc., rote counting is often taught or corrected through the use of little rhymes, as:

One, two, three, four, five,
I caught a hare alive;
Six, seven, eight, nine, ten,
I let him go again.

And,

One, two,
Buckle my shoe;

Three, four,
Shut the door;

Five, six,
Pick up sticks;

Seven, eight,
Lay them straight;

Nine, ten,
I see a fat hen.

Or,

One little, two little, three little Indians,
Four little, five little, six little Indians,
Seven little, eight little, nine little Indians,
Ten little Indian boys.

When a child can call the names of the numbers in order, the next step is to teach rational counting; that is, through counting objects about the room, the child must see the meaning of the names he has learned.

In having a pupil count objects, the teacher should take care that he doesn't get the idea that the second object is "two," the third one "three," etc. This is avoided by presenting groups and raising the question of "how many" and then finding it by counting. Thus, presenting three things as | | | ask, "How many?" The child counts, "One, two, three," and says, "There are three of them." Presenting four things, ask, "How many?" — "one, two, three, four, — there are four of them," etc. Make use of objects in the room. Ask how many windows, how many children, how many pictures, how many books, etc., until the children can apply their

rote counting to finding the "how many" of the things about them. Use objects in which the children are interested, not those made for the occasion, as splints, beads, etc.

The Names of the Figures

Before teaching any of the primary combinations, the pupil should recognize the names and meanings of the figures. Thus, he must associate the figure with the name and with the objects, as | | | | |, five, 5; ⁚ ⁚ ⁚, six, 6; etc. The teacher will devise many interesting ways of presenting this so as to fix these relations. One very enjoyable way to fix them is through a little "matching game." To play this game, take thirty cards — ten of them containing the figures, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (one figure on each card); ten others containing the names, one, two, three, etc.; and ten others containing objects as dots arranged as on domino cards. These cards are distributed among the pupils and, as a pupil is named, he runs to the front of the room and shows his card. The other pupils "match" the card, by coming up and standing in line. Thus, if a pupil shows "six," it is matched by 6 and ⁚ ⁚ ⁚, shown below.

In this and other ways, a pupil is soon able to associate the figures with the names and with the things which they represent.

Counting to Twenty and to One Hundred

Since our number system is based upon a scale of ten, counting by ones to ten is the first unit of instruction in counting. This is followed by counting to twenty.

3	13
4	14
5	15
6	16
7	17
8	18
9	19

Counting from 10 to 20 may be accomplished at the same time that the written forms are given. In this way, the child sees the similarity between 3 and 13, 4 and 14, etc. At the same time, he gets the sound "three," "thirteen"; "four," "fourteen"; etc., and sees that by adding "teen" the order is like the order he already knows.

Eleven and twelve must be taught separately, and there are no sound-aids to help the memory as in "teens."

The next step is counting by tens to one hundred. This is easily accomplished by calling the child's attention to the similarity in sound between "two" and "twenty"; "three" and "thirty"; "four" and "forty"; "five" and "fifty"; etc. Thus he sees that it is very much like counting by ones to ten if the suffix *ty* is annexed to each of the number names from two to nine. This, too, may be accomplished by using the written forms as in the "teens."

2	20
3	30
4	40
5	50
6	60
7	70
8	80
9	90

The third step is the filling of each decade with the one, two, three, etc., as twenty-one, twenty-two, twenty-three, etc.

Reading and Writing Numbers

When a child is first taught to read and write numbers, the place-value feature of our system of writing numbers should not be emphasized even if it is taught at all. He should be taught to read and write numbers as he reads and writes words. Thus, while writing 36, say, "this is thirty-six," etc. The child thus sees that the 3 suggests the thirty and the 6 the six, and easily reads any two-figured number. In the same way he is taught to read hundreds, thousands, etc. The real significance of "place value" should be reserved until the child is more mature and until such knowledge is needed in order to understand the *why* of the written processes.

CHAPTER IV

THE PRIMARY FACTS OF ADDITION: WRITTEN ADDITION

ASIDE from a few primary facts that have been picked up incidentally during the work in counting in the first year, the work of teaching the primary facts seems to be more economically done by teaching but one process at a time, rather than by teaching all processes about a given number at once, as in the Grube plan. While teachers differ as to the best order in which to teach the primary facts of addition, whether to teach the tables of ones, twos, threes, etc., or to teach all of the addition facts about a certain number, the order is much less important than the nature of the drill work that is needed to make the facts automatic. However, the order *is* important enough to merit serious thought and discussion.

THE FIRST GROUP OF PRIMARY FACTS

There are forty-five possible combinations of two one-figured numbers. These are called "the forty-five primary facts of addition."

The first group of these facts to be thoroughly fixed, whatever order is used, consists of the twenty-five combinations whose sums do not exceed ten. This is not

only the logical and psychological division of the facts, but an economic one in the matter of learning them. The most economic order seems to be to follow the tables in the ones and twos at least; for, when a child can count rationally by ones, he really knows nine of the first twenty-five facts; that is, he sees that to add one is to call the next number in the order of counting. However, some drill in calling these sums when seeing the written figures is necessary.

When any new sum is learned, the pupil should see it written. This written form should be placed before him for days until it leaves a mental picture. Thus, when the ones are presented, the pupil should see:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

This should leave such a mental image that when seeing

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

he can automatically name, or write, the sum. Each combination learned must be recognized instantly when written in either of the two possible forms shown below:

$$\begin{array}{cc} 3 & 1 \\ 1 & 3 \\ \hline \end{array} \quad \begin{array}{cc} 4 & 1 \\ 1 & 4 \\ \hline \end{array} \quad \begin{array}{cc} 5 & 1 \\ 1 & 5 \\ \hline \end{array} \quad \begin{array}{cc} 6 & 1 \\ 1 & 6 \\ \hline \end{array} \quad \begin{array}{cc} 7 & 1 \\ 1 & 7 \\ \hline \end{array}$$

In order to see the meaning of addition, the pupil should find the first group of twenty-five facts through counting objects. Thus, in the table of twos, 2+2,

3+2, 4+2, etc., to find 4+2, he should take four objects, then picking up two others, count "four, five, six" and write $\begin{array}{r}4\\ \underline{2}\\ 6\end{array}$. In this way he should discover for himself the facts of each table up to 5+5.

The Use of Objects

Objects are used to fix the *meaning* of addition, not to fix the facts themselves; that is, through this objective discovery of the facts, the pupil sees clearly what is meant by "three and four are seven," and is able to use these facts in applications to problems within the range of his experiences. When these facts have thus been rationalized, the objects have served their purpose and their use should be discarded. In the drills that follow, if a pupil miscalls a fact, the right result should be shown or told him at once. Nothing is gained by having him find the result again through counting. In fact, to have him do so is to establish a "counting habit" that must be broken up later. If 6+3 is given incorrectly, 6+3=9 should be shown at once.

The Order and Nature of Drills

The drills must build up a mental imagery in the child's mind. Hence, the first drills should be sight drills; that is, drills in which the child sees the figures. In these drills the figures should be written in column form as they are to appear later in written work.

These sight drills form a mental imagery so that in

pure mental drills, drills in which the figures are not seen, the child calls results from this mental picture. There are some children, however, who learn the combinations more easily by hearing and repeating the facts as "five and three are eight"; hence, this kind of drill should also find a place, but not as important a place as the sight drill.

The Importance and Extent of Drills

The pupil must have automatic control of all the primary facts and all the related facts needed in the fundamental processes. It is not sufficient that he knows the meaning of "four and five" and can find it through counting, or even recall it after a moment's time; but he must know it automatically. Just as he calls the words in a reading lesson without consciously recalling the letters of each word, so he must be able to call sums, products, etc., without being conscious of the numbers represented by the figures.

The drill work necessary to attain this need not be dull and repressing in the least. It may be made the most enjoyable part of the work. An appeal to the play instinct, the use of scoring games, flash cards, charts, number downs, races, playing store, and various games stimulate a keen interest in the work. This subject is discussed in Chapter VIII.

Preparing for the Second Group of Primary Facts

Before proceeding beyond ten, all the subtraction facts corresponding to the first group of addition facts must

be made automatic. The methods and devices used in teaching the subtraction facts are given in Chapter V.

The success of the method given below for teaching the remaining twenty facts of addition requires automatic control of the following subtraction facts:

10	10	9	8	8	7	7	6	6	5	5	4	4	3	3	2
9	8	1	1	2	1	2	1	2	1	2	1	2	1	2	1

This will be evident from the method shown below.

The next step in the preparation is to bring out the meaning of "teen." That is, show the pupils that thirteen, fourteen, fifteen, etc., mean three and ten, four and ten, five and ten, etc.

Let the pupils then tell at sight

10	10	10	10	10	10	10
9	8	7	6	5	4	3

Let $10+1=11$ and $10+2=12$ follow this work, since the sums are not called "oneteen" and "twoteen," but eleven and twelve.

Development of the Remaining Twenty Facts

By the plan here presented, the pupil makes the remaining tables much more easily than through counting, and by a method much more useful to him as a means of retaining the facts. In fact, to find through counting that eight and nine are seventeen in no way aids the memory in retaining this fact. The child is unable to recognize the number of objects in such large groups, so the finding of the fact in this way has no advantage whatever in fur-

nishing a mental imagery to aid the memory. The purpose of finding the first group through counting was not to help fix the facts, but to make clear the meaning of addition. But addition should be thoroughly clarified by this time and the objects should have served their purpose.

Begin the remaining twenty facts with the "nines." Write upon the blackboard the following:

9	9	9	9	9	9	9	9
9	8	7	6	5	4	3	2

Now take some one of these and, by the use of objects, show that one less than the number added to nine is the number of "teen" in the sum. Thus:

Through drill, train the pupils to form the habit of looking at the number to be added to 9 and thinking "one less" as the number of "teen." Thus pointing to the numbers written below, say "Add to nine."

5 7 6 4 8

As the teacher points to them, the pupils subtract one from each and say: "fourteen," "sixteen," "fifteen," "thirteen," "seventeen." The pupil is now ready to drill upon the tables as written above.

Take up the table of "eights" in the same way showing

that two less than the number added to 8 shows the number of "teen" in the sum.

Using charts, flash cards, games, playing store, etc., as in the first group, drill until the "eights" and "nines" can be given automatically.

They are the following combinations:

$$\begin{array}{ccccccc} 9 & 9 & 9 & 8 & 8 & 9 & 8 \\ \underline{9} & \underline{5} & \underline{8} & \underline{6} & \underline{8} & \underline{7} & \underline{7} \end{array}$$

$$\begin{array}{ccccccc} 8 & 9 & 8 & 9 & 8 & 9 & 9 \\ \underline{5} & \underline{3} & \underline{4} & \underline{4} & \underline{3} & \underline{2} & \underline{6} \end{array}$$

It will be found that these fourteen facts, usually the most difficult to teach, are as easily fixed as the simple facts of the first group.

There are yet six facts to teach. They are:

$$\begin{array}{cccccc} 7 & 7 & 7 & 7 & 6 & 6 \\ \underline{\ 7} & \underline{\ 6} & \underline{\ 5} & \underline{\ 4} & \underline{\ 6} & \underline{\ 5} \\ 14 & 13 & 12 & 11 & 12 & 11 \end{array}$$

These may be given upon cards and the pupil asked to find through counting that they are correct. The chief thing, however, is not the finding that they are correct, but the fixing of them as mental pictures.

Adding Zeros

While to add zero is to add nothing at all, and while the combinations with zero are not among the "forty-five primary facts" of addition, some practice in calling such

combinations is necessary in order to give an automatic control of them. Such combinations, however, need not receive as much time as the other facts.

A Complete Chart of Primary Addition Facts

The child needs drill upon all the possible ways of writing the figures including the zeros. Thus, there are "one hundred facts" instead of the so-called "forty-five facts" when zeros are used. In the following table, the easiest group is given first and the hardest last. Do not spend as much time, then, with the first part of the table as with the last.

0 0	0 3	4 0	1 0	6 0	0 8	0 1	7 0	0 5	3 0	9 0	5 0	2 0	0 4	8 0	0 6	0 9	0 2	0 7	1 1
1 2	1 8	4 1	1 5	7 1	6 1	1 3	2 2	1 9	1 6	1 7	2 1	3 3	1 4	8 1	4 4	3 1	5 5	5 1	9 1
2 3	3 4	2 6	4 2	3 5	8 2	2 5	6 3	7 2	4 3	3 2	5 2	7 3	2 4	2 7	5 3	6 2	3 6	2 8	3 7
9 2	4 7	6 6	4 6	4 8	8 3	7 7	9 4	4 5	3 9	6 4	2 9	8 4	3 8	4 9	9 3	7 4	8 8	5 4	9 9
6 7	9 7	5 7	7 8	6 8	9 6	5 6	8 5	9 8	5 9	7 5	7 6	8 7	6 5	8 6	9 5	7 9	5 8	6 9	8 9

Such a chart as this should be made for permanent use and also a hundred flash cards containing the same numbers should be made. Daily drill from these for a few minutes should be kept up long after they are "learned" and when other facts or processes are being developed.

Drills that Prepare for Written Work

Ability to give these primary facts automatically is not the only ability required in adding a column. Thus, to add the column in the margin, if we add down, 9+6 is the only primary combination. The next is 15+8, the next is 23+4, and the next is 27+7. Thus it is plain that to add columns the pupil must be able to add mentally (without seeing the figures) a two-figured number to a one-figured number. While this is closely related to the primary facts, special drill upon such combinations is needed. Such drills are sometimes called "adding by endings." Thus 8 added to 9 gives 17, a number ending in 7; so 8 added to any number ending in 9 gives a number in the next decade ending in 7. While there should be some sight work from charts of such combinations as

$$\begin{array}{r}9\\6\\8\\4\\\underline{7}\end{array}$$

$$\begin{array}{rrrrrrrr}7 & 27 & 47 & 37 & 97 & 67 & 17 & 57\\ \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8} & \underline{8}\end{array}$$

$$\begin{array}{rrrrrrrrr}9 & 29 & 19 & 59 & 89 & 39 & 79 & 49 & \\ \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{6} & \underline{\text{etc.}}\end{array}$$

for all of the primary combinations, a great deal of pure mental work should be done in which the teacher should point to any one of the nine digits

1 2 3 4 5 6 7 8 9

and say, "Add 17," "Add 26," "Add 58," "Add 43," for all numbers from 10 to 100.

Adding Single Columns

Before pupils are required to add two or more columns, they should have some skill in adding single columns, for adding numbers of two or more places requires, in general, carrying and also requires a greater tax upon attention; that is, longer attention before a break of taking up a new exercise.

A great deal of the drill work should be sight drill from charts, so that the effort of recording the figures in the answers will not detract from the attention needed for the addition. It is because pupils are plunged from the table of primary facts, not yet made automatic, directly into written addition, that they develop slow, inaccurate habits; and, instead of really adding, they find the sums through counting. When a pupil is slow and inaccurate in written work or has the "counting habit," there has not been sufficient drill in developing one or more of the three fundamental abilities: (1) automatic control of the forty-five primary facts; (2) automatic control of adding mentally a two-figured number to a one-figured number; and (3) the habit of reading a sum from a single column.

Written Work in Addition

There is a question among teachers as to whether the written processes should be rationalized or whether the pupil should merely form the correct habit of doing them. Since the fundamental processes are always performed in the same way and arise so often in all future work,

rationalization need not be urged as a help to the memory. Also, the efficient use of the processes in the solution of problems depends upon a rationalization of the *meanings* of the processes, not upon the rationalization of *modes of procedure* — carrying, borrowing, etc. — hence, it seems that there is little argument for rationalization. Upon this one principle all will agree, viz.: The important and essential thing in the fundamental processes is that the pupils form correct habits in the modes of procedure; that is, habituation, and not rationalization, is of chief importance.

There are pupils, however, to whom the work is made more interesting through rationalization and there are those who always want some authority for doing what they do. Rationalization need not take much time. Even if all of the class do not get clearly the "why," the objective presentation, or some method of rationalization of a process, satisfies the pupils that there is a reasonable basis for proceeding with the work as we do, and to some it makes the work more interesting.

Method of Development

Before the process of addition can be rationalized, the decimal-place-value feature of our notation must be understood. This can be shown by splints, toothpicks, or something of this nature. First write upon the board two or three two-place numbers as 24, 35, 46. Have the pupils count as many splints by ones. Then count ten and put a rubber band around them, then another ten in the same way, and so on until less than ten are left.

If 24 is the number counted, the pupil finds 2 tens and 4 ones left. If 35, then he finds 3 tens and 5 ones. Thus he sees that the 2 of 24 stands for tens and the 4 for ones, etc. Now, to test the understanding of the pupils as to what you have presented, have a pupil hold up any number (up to ten) of tens, and a number of ones and have the rest of the class write down the number shown. Thus, if three tens and six are shown, the pupils write 36. In this way it is easy to show the decimal-place-value principle of our notation.

Now there are two principles involved in adding numbers of two or more places. *First,* only like numbers can be added. Hence, the numbers are so written that the *ones* will all be in one column, the *tens* in another, etc., and each column is added separately because it is made up of like things. *Second,* when the sum of any column is greater than ten, it is reduced to units of a higher order, and those of the higher order added to the next column. Hence, the first problems should include but the first of these two principles in order to introduce but one difficulty at a time. However, it will be necessary to take but a few problems in which there is no carrying, for it does not take long to form the habit of adding a column at a time and putting the result under the column added. The more difficult habit to fix is that of working from right to left, for in all other work the child works from left to right.

Then, to rationalize the process of addition, suppose you have written the problem given in the margin. With splints or toothpicks bound up in bundles of ten each

and with loose ones, have a pupil hold before the class 2 tens and 6 ones, and another pupil, 4 tens and 3 ones, to represent the 26 and 43. Then let the class tell how many both pupils have, and thus see that the 69 stands for the 6 tens and 9 ones held by the two pupils. This need not be presented objectively more than two or three times. The presentation should then be followed with a few days' drill in adding two or three numbers of two or three places in which there is no carrying. All drill should be under the direct supervision of the teacher so as to insure the forming of proper habits. As soon as pupils automatically work from right to left — that is, add ones first, then tens, etc. — they are able to take up the next class of exercises.

$$\begin{array}{r} 26 \\ \underline{43} \\ 69 \end{array}$$

This is done by taking up some such problem as the one in the margin and proceeding as before. Here the class sees that there are 15 ones. Have them separate the 15 ones into 1 ten and 5 ones. Have the class see, then, that instead of 7 tens and 15 ones, we have 8 tens and 5 ones. But few objective illustrations are needed to rationalize the carrying process. Drill should then follow until the process is fixed. It is a waste of time to present the work objectively except when it is first taken up. Do not require pupils to "explain" their problems daily by the use of splints. They have seen that there is a rational basis for carrying, and that is sufficient. In fact, even this much of the process of rationalization is unnecessary to the efficient use of the subject.

$$\begin{array}{r} 38 \\ \underline{47} \\ 85 \end{array}$$

Checking the Work

The best time to develop the habit of checking the computations is at the time the processes are first taught. The pupils should feel that when a process is performed but once the work is but half done. By reviewing it in some way he must become convinced of its accuracy.

In addition, it is best to form the habit of first adding in a given direction and checking by adding in the opposite direction. This brings up different combinations. Thus, in the exercise in the margin, by adding down, the combinations are 15, 23, 27; and 9, 13, 22, 25. By adding up, they are 12, 19, 27; and 5, 14, 18, 25. Until the pupil is pretty sure of the result, the first result should be kept on a bit of "scratch paper" until it has been verified by the check.

$$\begin{array}{r} 78 \\ 47 \\ 98 \\ \underline{34} \\ 257 \end{array}$$

Drills in Written Work

Skill comes through much practice. Besides drill in adding numbers of two or more places, sight and mental drill in the primary facts, adding by endings, and adding single columns should be continued. But little of the drill work should come from problems. There is a means of securing greater interest than that gained through problems, however real they are. It is the interest that comes to the pupil from the knowledge that he is developing greater skill daily. In order to show this clearly, exercises of the same weight should be used as a test very frequently; that is, exercises in adding four or five

numbers of two or three figures each should be given two or three times a week, throughout the term, and a record kept of the average number tried by the class and the average number that are correct. Graphs of these results kept in sight will prove an inspiration to the class, for, if the drilling is done properly, there will be a marked improvement shown during a month or a term. Individuals should also keep their own records in the same way.

Not only will pupils be interested to see the graph of their progress gradually rise, but they will also be interested in watching the graph of accuracy approach the graph of the number of exercises tried.

It will be found that the incentive from trying to beat one's former record and a desire to see the "graph of progress" rise will have a greater appeal to the pupils of the fourth grade and above than to the pupils of the lower grades.

Such a record as that described above will also serve another very important purpose. It will enable the teachers to standardize their work. Through the average of a number of such records, one will have a standard by which he may know what a child of a certain grade should do. In other words, when a child can perform but a certain number of exercises in a certain time with a certain per cent of accuracy, it is clear that in the particular subject he belongs in the third, fourth, fifth, etc., grade.

Another device to secure interest in classroom drill in written work is to have the pupils call out the order in which they finish their work. Thus, the first pupil

to finish an exercise calls "one" and writes "1" above his work; the next one calls "two" and writes "2" above his work; and so on. When all are done and the answers given, all who have wrong answers draw a line through their numbers. At the end of the period, if one has a record like 1, 1, 3, 4, it shows both speed and accuracy. If one has ~~15~~, ~~20~~, ~~19~~, ~~16~~, etc., it shows the teacher at a glance that the pupil is both slow and inaccurate. The bright pupils like such a contest, and the dull ones work hard when they know such a test is to be given once or twice a week.

The same device may be used in any kind of written work throughout all grades.

The Nature and Use of Problems

The problems of the primary grades are given chiefly for one of two purposes. They are given either to clarify the processes or to furnish a motive for, or interest in, the work. Hence, they must be very concrete and real to the pupil and must be those to which he might wish to know the answer. While enough problems should be used to show a need of arithmetic, when the purpose of the instruction is to develop skill in computation, pure abstract drills should be used. It is much easier, too, to make well-graded drills that will emphasize just the facts or processes needed than to get the same drills through problems, for a problem should present real conditions. Hence, the size of the numbers is regulated by the nature of the problems, and such drills are not in general as well graded as the pure abstract exercises.

CHAPTER V

THE PRIMARY FACTS AND PROCESSES OF SUBTRACTION

The primary facts of subtraction come directly from those of addition. Thus, when a child can tell you that 3 and 5 are 8, the question, "3 and how many are 8?" comes not as a new fact, but from the addition fact already known. The first drills should be answers to such questions as the one above. In sight drill they are written as follows:

*	3	4	*	*	4	6	*	*	7	
6	*	*	5	6	*	*	8	6	*	etc.
9	7	8	7	8	7	7	10	10	9	

The pupil gives the missing number that, with the given one, makes the sum below.

By subtraction we find the answer to three types of questions illustrated by:

(*a*) 5 apples taken from 8 apples leave how many apples?

(*b*) How many apples added to 5 apples make 8 apples?

(*c*) If one plate contains 8 apples and another 5 apples, how many more in the first plate?

That is, *Subtraction is the process of taking one number from another to find how many remain; of finding what number must be added to a given number to make a given sum; and of finding the difference between two numbers.*

While all these meanings should be shown objectively, the pupil must see clearly that the primary facts come from addition.

The nature of the drill work upon the primary facts depends upon the method to be used in written work. If the "addition method" is to be used, the thought in the drills for $14-9=5$ will be "9 and 5 are 14"; while, if the "taking-away method" is to be used, the thought will be "9 from 14 leaves 5."

The Addition Method

In deciding upon a method to use, the questions asked are: "By which method is there the least liability of error?" "Which method is more easily presented?" and, "By which method can the work be done most rapidly?" There are advocates of the "addition method" who would answer, "The addition method," to each of the above questions. However, there do not seem to be available data to warrant such a reply. But that there are strong points in favor of the method, one must recognize. Thus, it follows so closely the work of addition that much of the skill in addition is carried over to subtraction. It is also more nearly the method used by clerks in counting out change.

The way in which to present the addition method is first,

to take a problem in which each figure of the subtrahend represents a number less than the corresponding number in the minuend. Thus, to present the problem in the margin, the pupil must have acquired the idea from the drills upon the primary

$$\begin{array}{r} 48 \\ \underline{23} \\ 25 \end{array}$$

facts as, $\begin{array}{r} 9 \\ \underline{4} \\ 5 \end{array} \quad \begin{array}{r} 8 \\ \underline{3} \\ 5 \end{array} \quad \begin{array}{r} 6 \\ \underline{2} \\ 4 \end{array}$ etc., that the top number is the *sum* of the other two. Like "a corn on his chin" of Riley's *Man in the Moon,* which is "a dimple turned over, you know," so subtraction is "addition turned over, you know." Then the thoughts in the above problem are: 3 and *5* are 8, write 5; 2 and *2* are 4, write 2. Since the tables should have been made practically automatic before taking up the written work, there is practically nothing new to teach, for the pupil has the habit of beginning at the right-hand column and of considering one column at a time, which was formed in his work in addition. The "why" of this need not be shown. However, if addition was rationalized, the reason for this follows; but the work may be presented objectively if it is thought that such a presentation will answer the curious or make the work more interesting. Remember that the important thing is to establish proper habits of procedure and to develop skill in doing the work accurately and rapidly.

To develop the second problem, the one in which the minuend has digits less than the corresponding digits of the subtrahend, review those primary facts of subtraction whose minuends are two-figured numbers, as:

10	11	18	16	13	14	13	15	13	12	
8	7	9	9	8	6	7	6	9	8	etc.

Call attention to the fact that in these, the number below is less than the one directly above.

Then present some such example as the one in the margin. Call attention to the 4 being less than 6, so it must be the 4 of 14, which is the sum of 6 and some number, as in the table above. So ask "6 and what make 14?" Write the answer, 8, below the 6. Now, since 54 is the sum of 26 and some number, the 4 is the 4 from 14, the sum of 6 and 8; hence, the 1 (ten) of 14 was carried to 2 of the given addend. So the next question is "3 and what make 5?"

$$\begin{array}{r} 54 \\ \underline{26} \\ 28 \end{array}$$

There are two things, then, for the pupil to observe:

(1) When the number above is less than the one below it, picture it as the right-hand figure of some number of "teen."

(2) In such cases, *one* is always added to the next higher digit of the addend (subtrahend).

In the second example in the margin we think 8 and *3* are 11; write 3. 1 (carried from 11) and 6 and *5* are 12; write 5. 1 (carried from 12) and 2 and *2* are 5; write 2.

$$\begin{array}{r} 521 \\ \underline{268} \\ 253 \end{array}$$

If "why" arises, show that it is merely the carrying that was done in addition, for the top number is the *sum* of the given number and the number found.

In presenting this, or any other process, the process should first be presented very carefully by the teacher

and enough examples taken to show fully the "how." Then the pupils should be sent to the blackboard where the first work can be done directly under the eye of the teacher in order that she may detect any mistake in using the process before that mistake becomes a habit. If seat work or home work is not given until every pupil knows how to perform the process, then the only errors made will be the accidental errors. The purpose of further drill, seat work, home work, etc., is to develop greater skill in handling the problems.

The Taking-away Method

The method in most common use in the schools in this country is the "taking-away" method of subtraction. There is no trouble in presenting the first class of problems; namely, those in which each number in the subtrahend is less than the corresponding number in the minuend. Thus, in the problem in the margin, the pupil thinks 6 from 8 leaves 2; write 2. 1 from 4 leaves 3; write 3. Since the pupil knows the tables automatically before taking up written work, this problem presents nothing new, for the habit of subtracting one column at a time and of working from right to left follows the habit established in addition and is not really a new habit here. After a few problems to see that these habits are fixed, further drill need not be given before taking up the next class of problems; that is, those in which the so-called "borrowing" is necessary. The pupil is very apt to ask *why* here, so the teacher may

$$\begin{array}{r} 48 \\ \underline{16} \\ 32 \end{array}$$

well present one or two examples objectively. Thus, to present the problem in the margin objectively, show 62 as 6 bundles of tens and 2 ones, using toothpicks or splints. Now ask a pupil to take away 7 ones. He sees that, before he can do this, he must have more than 2 ones. Therefore, have him take one bundle of the 6 tens; take off the band and put the 10 splints with the ones, thus giving 12 ones. Now taking away 7 leaves 5; write 5. Now ask him to take away 2 tens from the remaining 5 tens, leaving 3; write 3. After a few objective presentations and a few exercises without objects, the pupils should be sent to the blackboard and should work under the direct eye of the teacher until they can work with small numbers with but slight danger of error. Increase the difficulty of the exercises very gradually. Avoid exercises in which it is necessary to take one from the next higher order when the figure of that order is zero, as in 502 − 136, until the pupils handle the other simpler exercises well. Next take up such an exercise having zeros in the minuend, as a special lesson. Thus, in the problem above, show that one is taken from 5, making 10 tens; and that one of the tens is then taken, leaving 9. This class of exercise presents special difficulty, as may readily be seen, and requires special drill. This difficulty, however, does not occur in the use of the addition method.

$$\begin{array}{r} 62 \\ \underline{27} \\ 35 \end{array}$$

CHAPTER VI

THE PRIMARY FACTS AND PROCESSES OF MULTIPLICATION

The Tables Developed from Addition

Multiplying by a whole number is only a short way of finding the result of adding a number of equal numbers. Thus, 7×345 is only a short way of writing and finding $345+345+345+345+345+345+345$. Hence, it would seem that the most logical and psychological method of developing the primary facts is through addition, and experience shows this to be the most economic and efficient method.

Begin by selecting all the doubles from the forty-five facts of addition. Show the class the following drill cards used in addition:

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
—	—	—	—	—	—	—	—	—

The results, of course, are known. Now remove the cards and ask the class to tell you how many of each number they saw on a card and to tell you the result. The reply will naturally be, "two ones are two," "two twos are four," "two threes are six," "two fours are

eight," etc. Hence, the only new thing in the "two times" table is the notation. That is, $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 3 = 6$, etc., which should be read, "two ones are two," and not "two *times* one *equals* two." Always use language that gives the clearest picture of the real meaning of the process.

Before taking up multiplication, the pupils should have developed some skill in column addition. They should at least be able to add single columns of four or five numbers. Then to develop the "three times" table, meaning three of each number have been added, have the class write down three of each of the nine digits as follows and find the sum. Thus:

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27
\end{array}
$$

When the pupil sees what a time-saver this new process of multiplication is, he will have a motive for learning the tables and how to use them. Thus, to find the sum in the margin, let him see that by remembering that three 8's are 24, he does not need to add. Also, show him that the second method in the margin is a shorter way to write the same thing, and that he now thinks three 8's, three 6's, and three 4's, instead of *seeing* them, as in addition. Thus, by this method of development the written work can be taken up with each table, thereby forming

$$
\begin{array}{r}
468 \\
468 \\
\underline{468}
\end{array}
\qquad
\begin{array}{r}
468 \\
\underline{\times 3}
\end{array}
$$

a strong motive for learning this new process and also furnishing much needed drill in fixing the facts.

After each table is developed by addition, it is written down as multiplication, thus:

$3\times1=3$	$3\times4=12$	$3\times7=21$
$3\times2=6$	$3\times5=15$	$3\times8=24$
$3\times3=9$	$3\times6=18$	$3\times9=27$

These are read "three 1's," "three 2's," "three 3's," etc., for this is just what the pupil sees in the development.

In many textbooks the tables are written:

1×3	3×3	5×3	7×3	9×3
2×3	4×3	6×3	8×3	10×3

This is because the tables were developed through counting by 3 and thus the notation accurately describes the thing the pupil saw in the development. This is the table of "threes" and not the "three times" table, which was developed through addition.

After a few tables have been found, the word "times" may be used, for the expression will now have a meaning when the pupil knows that 3×5 means that three 5's have been added.

This method of development is to make clear the meaning of multiplication, that it is a short process to save adding equal addends. Hence, it is not important that all of the tables be found by the pupil. Perhaps the finding of the tables through the "five times" table is sufficient to fix the full meaning and use of multipli-

cation. If the child's interest seems to lag in the making of the tables after the first four or five tables have been found, the "nine times" table may well be taken up next as follows: The pupil should have been shown objectively or otherwise from the first that $2 \times 3 = 3 \times 2$, $3 \times 5 = 5 \times 3$, etc. Thus in the diagram in the margin, in rows there are 5×3, while in columns there are 3×5.

Now, since $2 \times 9 = 18$, $9 \times 2 = 18$; since $3 \times 9 = 27$, $9 \times 3 = 27$; since $4 \times 9 = 36$, $9 \times 4 = 36$. These are known from the tables already learned. Now, point out that in these three products in the "nine times" table, the tens' digit of each product is just one less than the number multiplied by 9, and that in every case the sum of the digits in the product is 9. Now tell the class that this is always so, and hence to find 9 times any number, as 9×6, it is merely necessary to write one less than 6 for the tens' digit, that is, 5, and for the ones' digit write a number that with 5 makes 9. That is, $9 \times 6 = 54$, $9 \times 8 = 72$, $9 \times 5 = 45$, etc.

Now, if the 3 times, 4 times, 5 times, and 9 times tables have been developed, but six new facts remain. They are: 6×6, 6×7, 6×8, 7×7, 7×8, and 8×8. It is better to give these facts to the child than to have him find them through addition; or the facts may be given and the pupils may prove them by addition. This will be a test as to whether or not they really understand the meaning of multiplication.

Drill must be continued on the tables until all the facts can be recalled automatically.

A Second Method of Development

Many of the older textbooks and some of the newer ones develop the tables through counting by 2's, 3's, 4's, 5's, etc. The plan is to learn to count by 2's, 3's, etc., then build up the tables as follows:

								2
							2	2
						2	2	2
					2	2	2	2
				2	2	2	2	2
			2	2	2	2	2	2
		2	2	2	2	2	2	2
	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2
2	4	6	8	10	12	14	16	18

While, of course, counting by 2's is adding 2's, yet to the child this does not seem to be addition as in the former method, and does not leave as clear a meaning in the child's mind of what multiplication really is. Moreover, the learning of a single table does not fit the pupil at once to take up written work as is the case in the former method. Thus, the pupil could not use the above table to find 2×768. Or, if the 3's were learned in that way, he could not find 3 times any number, as 3×645, from this table. There is a further danger, too, that pupils may rely upon counting to find the products in future work rather than fix the facts through memory.

Observe, also, that to use the method given here, the tables are not written in the same order as when developed by the first method. They would be written: 1×2, 2×2, 3×2, 4×2, 5×2, 6×2, etc. The difference in the two methods of development accounts for the two ways of writing the tables.

This method of development, then, is not recommended.

Drills that Prepare for Written Work

Just as in written addition it was seen that the mere learning of the tables did not fit the pupil for successful written work, so the tables alone are not sufficient for successful written work in multiplication. Thus, to find 3×485, the first product is but 3×5, but the next one is $3\times8+1$, and the next $3\times4+2$.

So there should be drill upon announcing any product plus any one-figured number. A simple device for such drill is a chart made as follows:

$$\left.\begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\\9\end{matrix}\right\}\times\left\{\begin{matrix}9\\8\\7\\6\\5\\4\\3\\2\\1\end{matrix}\right.+\left\{\begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\\9\end{matrix}\right.$$

To use the chart, merely point to any number whatever in each column, going from left to right. Then let the pupil give the product of the first two plus the third. It will be seen that no possible combination in any written work can occur that cannot be drilled upon from such a chart. A five-minute drill daily upon such a chart will aid very greatly in efficient written work.

Written Multiplication

It was shown above that written multiplication by multipliers of one figure should be taken up when each table is developed in order to furnish a motive for learning the tables and to give further drill that is needed to fix them. When a pupil sees that instead of a long problem in addition, as in the left-hand margin, the same result may be found by multiplication, as in the right-hand margin, he not only understands what multiplication means, but sees a very strong motive for learning it.

$$\begin{array}{r} 475 \\ 475 \\ 475 \\ 475 \\ 475 \\ 475 \\ \underline{475} \\ 3325 \end{array} \qquad \begin{array}{r} 475 \\ \underline{7} \\ 3325 \end{array}$$

In presenting written multiplication, it is necessary merely to point out that, instead of *seeing* the number of equal addends, we *think* them; that is, instead of seeing seven 5's as in addition, we think seven 5's are 35; instead of seeing seven 7's, we think seven 7's are 49, and 3 to carry are 52; instead of seeing seven 4's, we think seven 4's are 28 and 5 are 33. This is the only explanation necessary when the tables are taught by the first method given here.

When the Multiplier Has Two or More Figures

Before written multiplication by two-figured multipliers is taken up, the pupil should understand the effect of annexing a zero to a whole number. He should never be allowed to multiply by 10 by writing down the multiplier beneath the multiplicand, as is so often seen even in our best schools. The pupil should be shown that annexing

a zero to a whole number has the effect of moving each digit one order higher, and thus multiplies each digit by 10. Thus in 36, by annexing a zero making 360, the 6 ones become 6 tens, and the 3 tens become 3 hundreds. If place value has not been taught up to this time, it must be taught here if the pupil is to see *why* annexing a zero multiplies by 10. This may be followed by showing that to multiply by any number of tens, as 40, first multiply by 4 and then annex a zero to the product, which multiplies the result by 10. These two facts — that annexing a zero multiplies an integer by ten, and that any integer is multiplied by a multiple of ten, by multiplying it by the multiple and annexing a zero to the product — are preliminary to taking up two-figured multipliers.

Pupils in the third or fourth grade, where this work is first taken up, are too young to follow a full rationalization of the process. In order that the pupil may satisfy his curiosity as to the reason for performing the work as he does, the teacher may ask some such questions as the following: "Three 8's and two more 8's are how many 8's?" "Six 5's and three more 5's are how many 5's?" Then, "Five 83's and twenty more 83's are how many 83's?" The class should reply, "Twenty-five 83's." The teacher may then say, "We will find 5×83 and then 20×83 and add the results, and that will give us 25×83." As the work is performed by the teacher upon the blackboard, the pupils giving the products as she records them, she should point out that the first product (415) is 5×83. Now in finding 20×83, the teacher gets from the class

$$\begin{array}{r} 83 \\ \underline{25} \\ 415 \\ \underline{166} \\ 2075 \end{array}$$

that 20 is 10 times 2, and says, "Then if I multiply 83 by 2 and place the first result in tens' place, that will be multiplying the result by 10, so in that way I multiply by 10×2 or 20." It will be observed that this is a very imperfect rationalization, but it will be found that it is enough of the "reason why" to satisfy the curious and is perhaps as much as the class can comprehend, if not more.

As in all cases, the work is first presented several times by the teacher, then the class works at the blackboard under her direct supervision until the process is mastered and before any home or seat work is given.

CHAPTER VII

THE PRIMARY FACTS AND PROCESSES OF DIVISION

The Tables

The primary facts of division follow at once from the multiplication facts. Thus, ask such questions as, "Three 5's are how many?" (Answer, 15.) "How many 5's in 15?" "Four 6's are how many?" "How many 6's in 24?" "Three 7's are how many?" "How many 7's in 21?" 3×3-

Notice that these are the questions that should bring out an answer from the child's mental picture of multiplication. Thus, 4×8 means four 8's written and added, as in the margin. So, when the question, "How many 8's in 32?" is asked, the answer very naturally is "Four."

$$4 \times 8 = \begin{array}{r} 8 \\ 8 \\ 8 \\ \underline{8} \\ 32 \end{array}$$

The notations should then be given by saying, "This is the way we write the fact that there are four 8's in 32." As a drill to fix the meaning of the notations, have a large miscellaneous number of the facts with answers written as:

$$\begin{array}{r} 8\overline{)32} \\ \hline 4 \end{array}$$

$$\begin{array}{r} 3\overline{)21} \\ \hline 7 \end{array} \quad \begin{array}{r} 6\overline{)30} \\ \hline 5 \end{array} \quad \begin{array}{r} 8\overline{)56} \\ \hline 7 \end{array} \quad \begin{array}{r} 5\overline{)45} \\ \hline 9 \end{array} \quad \begin{array}{r} 7\overline{)42} \\ \hline 6 \end{array} \quad \begin{array}{r} 4\overline{)28} \\ \hline 7 \end{array} \quad \begin{array}{r} 6\overline{)24} \\ \hline 4 \end{array}\text{, etc.}$$

and ask the class to read them. They should say, "There are seven 3's in 21"; "There are five 6's in 30"; "There are seven 8's in 56"; etc. Do not allow the expression "goes into," so often heard, and so often pronounced, "gus-sin-tu."

When the notation is understood, have the pupils write out all the division tables in this way from their knowledge of multiplication.

It will be observed that the meaning of the division here presented is *measurement;* that is, the quotient shows how many times the dividend contains the divisor. When these are well known, the *partition* phase of division may be presented. This phase is shown as follows:

			5	6	8		
3	2	4	5	6	8	7	6
3	2	4	5	6	8	7	6
3	2	4	5	6	8	7	6
9	6	12	20	24	32	21	18, etc.

Write upon the blackboard several sets of equal addends as shown above. Now, if it has not been done before, show the class, dividing up some object as you do so, that one of the *three* equal parts of anything is *one third* of it; one of the *four, one fourth* of it; one of the *five, one fifth* of it, etc., and thus through the rhythm of it, they can at once tell you the name of one of any number of equal parts of anything.

Now, say, "What is one of the three equal numbers that make 9?" (Answer, 3.) Then, "What part of 9 shall we call 3?" (Answer, one third of it.) "What is

one of the four equal numbers that make 20?" (Answer, 5.) "What then shall we call 5?" (Answer, one fourth of 20.)

These questions are first asked about a large number of facts written upon the blackboard in addition form. Later they are asked and answered from the mental picture of the multiplication facts.

The Notation of Unit Fractions

Without any further discussion of fractions, except the limited meaning and use shown above, the written notations may be given:

Thus, one sixth is written $\frac{1}{6}$
one seventh is written $\frac{1}{7}$
one eighth is written $\frac{1}{8}$, etc.

The pupil is now ready to write the facts that he has just given orally as follows:

$\frac{1}{3}$ of $6=2$; $\frac{1}{4}$ of $8=2$; $\frac{1}{5}$ of $15=3$; $\frac{1}{6}$ of $24=4$, etc.

The pupil should be shown that the first form of expressing division may be interpreted as "a part of" when the divisor is an abstract number. Thus, $\begin{array}{r}3\overline{)6}\\ \hline 2\end{array}$ may be interpreted either as "There are two 3's in 6," or "One third of 6 is 2."

These are the two most important forms of expressing division, but for the purpose of brief written analysis the form $6 \div 3=2$ should be given.

Since the form for written work is the first one given $\left(\begin{array}{l}3\overline{)6} \\ \quad 2\end{array}\right)$, this is the form that should be used in all sight drills. It may be interpreted as either the "measurement" or the "partition" type of division.

A mere knowledge of the division tables is not sufficient for successful written work. Thus, to find $3\overline{)162}$, the first division requires not one of the primary facts, but the fact that there are five 3's and 1 in 16. As a result, it is seen that not only is drill upon the tables needed, but the pupil must drill upon every possible dividend from one times the divisor to ten times the divisor. Thus, if "the sixes" are being studied, a drill is needed upon some such chart as the following, including all numbers from 6 to 60 inclusive:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
1	20	38	50	29	7	49	42	11	36	55	24
2	17	48	6	57	34	16	23	58	56	15	60
3	40	9	41	10	46	37	52	33	18	59	19
4	28	39	30	51	22	26	13	53	45	12	44
5	47	21	8	27	35	14	43	31	25	54	32

In drilling upon the table, for example in column *B*, the pupil says, "6 and 2 remaining; 8; 1 and 3 remaining; 6 and 3 remaining; 3 and 3 remaining." A few minutes daily from such a chart will greatly aid in securing rapid and accurate written work in short division.

The First Written Division

The first written work may be taken up along with the tables and should proceed very gradually from the tables themselves. Thus, when the pupil writes $2\overline{)8}$, $2\overline{)6}$, $2\overline{)12}$, with quotients 4, 3, 6, etc., it is a very easy step to get him to take up such examples as $2\overline{)86}$, $2\overline{)48}$, $2\overline{)128}$, with quotients 43, 24, 64, etc., where there are no remainders after each division. The oral drills prepared the pupils to think $5 \div 2 = 2$ and 1 remaining; $9 \div 2 = 4$ and 1 remaining; $7 \div 2 = 3$ and 1 remaining, etc. Therefore, it is an easy step to show that the 1 remaining, used with the number that follows, is the next number to divide.

If it seems wise to rationalize the work more fully, it may be done objectively. For example, to show that $52 \div 2 = 26$, show 52 as five bundles of ten splints and two ones. Now, if the five tens are arranged in two equal groups, there are 2 tens in each group and 1 ten remaining. Now, this ten and 2 make 12 ones. Dividing 12 into two equal groups, there are 6 in each. So there are 2 tens and 6, or 26, in each of the two parts.

It will be seen, however, that this explanation is really based upon the "partition idea" of division, and not upon "measurement." One could not ask children, "How many times do 5 tens contain 2?" and expect an answer. So, if the work is to be presented objectively, more oral drill upon the partition phase of the subject is necessary. Thus, in giving $5 \div 2$, $7 \div 2$, $9 \div 2$, $11 \div 2$, etc., the pupil

says, "One half of 5 is 2 and 1 undivided," "One half of 7 is 3 and 1 undivided," etc.

In drilling upon the table given on page 64 the pupil thinks "One sixth of 20 is 3 and 2 undivided"; "One sixth of 17 is 2 and 5 undivided," etc.; and not "There are three 6's and 2 remaining in 20"; "There are two 6's and 5 remaining in 17," etc.

It seems unwise, however, to take up the matter of rationalization with pupils of the third grade unless they are exceptional children. It seems better to begin with very easy work as given above, and to show *how* to do the work and then, as the work becomes easy through drill, to progress to more difficult exercises.

Long Division

The pupil has been performing short division for nearly a year before taking up long division. By writing down all the work that was done mentally, the pupil should be shown the longer form.

Thus:

$$\begin{array}{r} 9\underline{)2412} \\ 268 \end{array} \qquad\qquad \begin{array}{r} 268 \\ 9\overline{)2412} \\ \underline{18} \\ 61 \\ \underline{54} \\ 72 \\ \underline{72} \end{array}$$

As this is shown, question the pupil as to what he *thought* in short division and *write it down* in long division. Thus, in short division he thinks, "There are two 9's in

24 and 6 remaining, for $2 \times 9 = 18$, and $24 - 18 = 6$." In long division this is written down.

The only reason that pupils find long division difficult is because they have trouble in estimating the quotient figures. This, then, is the feature upon which they should be drilled. However, as a preparation for long division, it is well to give a few lessons in which the divisors are multiples of from 2 to 9 times some power of ten, the work to be done by dividing both dividend and divisor by that power of ten and by using short division.

Thus: $3\cancel{0}\cancel{0}\cancel{0})\underline{24\cancel{0}\cancel{0}\cancel{0}}$ $\quad 4\cancel{0}\cancel{0})\underline{17\cancel{0}\cancel{0}}$ etc.

$\qquad\qquad 8 \qquad\qquad\qquad 4;\ 100$ rem.

The teacher should recognize that it is much easier to estimate the quotient figures from some divisors than from others and that some quotients are much more easily determined than others. Thus, the easiest divisors are those that are *nearly* some multiple of 10 as 61, 71, 81, etc.; or 59, 69, 79, etc., the child thinking as he uses them, "About 60, about 70, about 80," etc. The easiest quotients to determine are those that fall about midway between two consecutive multiples of 10, as 65 (midway between 60 and 70), 74, 75, 76, 84, 85, 86, 54, 55, 56, etc.

It is seen then that, as to difficulty, there can be at least three grades of long division exercises. The dividends should be made by selecting the divisors and quotients wanted. The types are 61×75; 59×69; 64×59. The first of these will have an easy divisor (61) and an easy quotient (75); the second an easy divisor (59),

but a hard quotient (69); and the third, both a hard divisor (64) and a hard quotient (59).

Thus, to find $6156 \div 81$, as shown by following the work in the margin, the only consideration is $61 \div 8 = 7$ and 5 remaining; $48 \div 8 = 6$.

$$\begin{array}{r} 76 \\ 81\overline{)6156} \\ \underline{567} \\ 486 \\ \underline{486} \end{array}$$

In $4209 \div 61$, as may be observed, the consideration is not merely $42 \div 6 = 7$, but a further observation that the next digit (0) of the dividend does not contain the next digit (1) of the divisor 7 times, and hence the quotient digit cannot be larger than 6.

$$\begin{array}{r} 69 \\ 61\overline{)4209} \\ \underline{366} \\ 549 \\ \underline{549} \end{array}$$

While in $2124 \div 36$, the pupil who makes the same kind of estimate as he made above will observe that $21 \div 3 = 7$ and try 6. But he must be shown that, before writing down 6 in the quotient, he should think $3 \times 6 = 18$; $21 - 18 = 3$; this 3 with the 2 makes 32; now, 32 does not contain 6 six times, hence the quotient cannot be larger than 5.

$$\begin{array}{r} 59 \\ 36\overline{)2124} \\ \underline{180} \\ 324 \\ \underline{324} \end{array}$$

These three illustrations should make it clear that there are at least three grades of exercises as to difficulty, even in exercises containing the same number of figures.

A large number of each class of exercises should be worked out and charts prepared as follows:

Type I

	A	B	C	D	E	F	G	H	
1	2025	2916	3645	4293	3807	4374	5103	5265	
2	6885	5994	6966	5427	6804	7695	6075	7614	÷81
3	3726	6156	7857	5184	4617	7723	7533	5913	

Type II

4	5551	4819	3111	4941	3599	5978	2562	5429	
5	5002	4758	3782	3538	4209	3172	4148	2379	÷61
6	2501	3721	4392	2928	5612	2989	2867	4331	

Type III

7	4788	7068	5544	4104	2052	5548	7372	5092	
8	4028	6308	3496	5776	4864	2888	6536	6612	÷76
9	2736	5624	7296	2812	2128	4332	3572	3268	

In type I, it is easy to estimate the quotient figure, for dividing the number represented by the first two figures of the dividend by the first number in the divisor *always* gives the correct quotient figure. This is not always the case in type II. A mental multiplication, however, will show whether or not this result follows. But in type III, more careful consideration is required.

There are two uses of such drill charts: (1) They may be used for "sight work" in which the pupil will give the first quotient figure only; and (2) they may be used for the usual written work.

CHAPTER VIII

THE USE OF GAMES IN NUMBER WORK

It is clearly recognized by all who have given any thought to the subject, that a very necessary condition for true learning is that the process be self-actuated through motive or interest. If the thing taught a child can be made to meet his needs or interests, he finds but little trouble in learning it. Hence, a problem of the teacher is to find a use to the child for the number facts that she must teach him. A number problem has no great interest to a child unless the answer to it serves some purpose — meets some personal need or appeals to his curiosity. The range of problems that have any real interest to a pupil learning the primary facts is so limited that they are not sufficient to hold the interest long.

Play is the chief occupation of most children of the lower grades, hence the appeal through the play instinct — games, dramatized vocations, etc. — is a very strong one, and may be used to very great advantage in the early number work.

The games may be grouped as to use as: (1) schoolroom games; (2) playground games; and (3) home games.

The schoolroom games must be those through which the teacher may get a maximum amount of number combinations before the class with as little of the time taken up by the game as possible. The games must be so selected as to have all pupils silently attending to the number combinations when not actually reciting.

The school ground and home games are more to furnish a motive for learning the work presented in class and, hence, the number work should in no way overshadow the recreative elements of the game. But such games serve to show the pupil that what he learns in school may be made use of out of school; and, by turning them into "make-believe" games, they may be used in the classroom drills, as will be shown in the games that follow.

As to their nature, both the in-school and out-of-school games may be grouped under three general heads: (1) scoring games; (2) imaginative or "make-believe" games; and (3) games of motor activity.

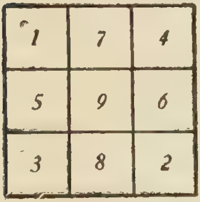
SCHOOLROOM GAMES

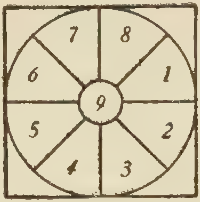
SCORING GAMES

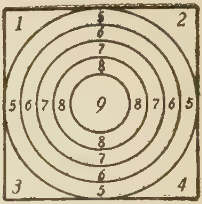
The scoring games are perhaps the most familiar to all. They include Bean Bag, Ring Toss, Ten Pins, and such standard games as may be found in any toy shop. Yet, as found in the toy shop, they bring in but few of the combinations.

The essentials of a scoring game are: (1) that so little skill is required to score that every "throw" will give a number for a combination; and (2) that the targets in-

clude all the nine digits. Below are targets that may be used in some of the games that follow:





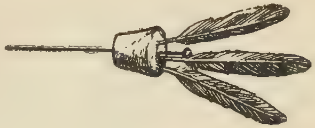


There are many tossing, bowling, and shooting games that may be easily devised by any resourceful teacher. The tossing, bowling, or shooting, however, does not have to be carried on to any great extent in the schoolroom. Any of the following scoring games may be used to score a few times, then the game turned into a "make-believe" game. Thus, the teacher may point to any number on such targets as these drawn above and pretend that they are the scores made by pupils, thus motivating the work through "make-believe" target practice.

Throwing the Arrow

A target about two feet across, like any of those shown, may be made on soft board and covered with burlap to hide the marks from the shots, and hung against the wall. The pupils throw an arrow made as follows: take a large pin or a slim finishing nail sharpened to a point, a small cork about one inch long and three fourths of an inch in diameter, and three feathers from five to seven inches long, and make an arrow as shown in the picture. This can be thrown with great accuracy.

Let each child throw twice and have the sum or product, depending upon the drill, of the numbers hit if he calls them correctly. After a few throws, the teacher may merely point to two numbers on the target, asking a pupil to name the score. If correct, it is added to his other scores. After a pupil can perform written addition, he may add his scores every five turns or chances and thus the same game may motivate the written addition.



Bean Bag

Take a large cardboard, or a light board, and make nine holes in it, numbering them from 1 to 9 inclusive. Leaning it against a support of some kind, the children toss small bean bags through the holes, scoring as in the arrow game.

Hook It

A board with nine hooks on it, numbered from 1 to 9 inclusive, is hung against the wall. The children try to ring the hooks by tossing soft rubber rings.

Bridge-board Marble Game

Cut nine semicircular archways about three inches in diameter from one edge of a board. Then support the board edgewise on the floor so that these arches are next the floor. Number the archways from 1 to 9 and let the children roll marbles through the openings, scoring as in the preceding games.

Bowling with Marbles

A target may be marked off on the floor and a marble rolled into it, the child scoring the numbers upon which the marble stops.

These are but a few suggestions as to the possibilities with scoring games. The methods of scoring may easily be varied.

IMAGINATIVE OR "MAKE-BELIEVE" GAMES

These include the dramatization of various vocations, as "playing store," "the mailman," "the milkman," and also the pretended playing of some scoring game, or other game, by use of diagrams drawn upon the blackboard.

There is no end to imaginative games that may be made by the teacher. The following will show how a resourceful teacher used a game, which she called "Playing Fireman," for several weeks without any loss of interest, and how the game suggested itself to her. The fire department had just passed the school ground during intermission and the children were greatly interested in watching it. When the period for number work came, the teacher asked how many would like to play "Fireman." Of course, all wanted to play. She stepped to the blackboard and drew two ladders leading to the top story of a burning building. Between the rungs of the ladders she placed the number combinations upon which she wished to drill. She said, "Now we will play that

the one who can run up one ladder and down the other without making an error has rescued some one from the burning building; but those who make a mistake have fallen down and so are not good firemen."

After a delightful period of "Playing Fireman" it was suggested that two permanent companies should be organized and that each company should have a fire chief. The pupils said that the strongest and boldest — quickest and most accurate — should be chosen chief; but, of course, each child hoped that he might be the one chosen for that office. Then followed more drill for a few days before having a contest to determine who should be the chiefs. At last, two were chosen. The chiefs wanted to choose their own companies. As a result, more drill was needed in order that the chiefs might see whom they wished to choose. The first day resulted in choosing but two or three for each side. Finally, all were chosen. The next step was for each chief to train his men for a special exhibition of skill. A day was fixed for the exhibition. Then each man in each company gave all the combinations. The time required and the errors made were noted in order to tell which company won the prize.

Then there was a big fire in a down-town district and whole blocks were aflame and many ladders were put up to rescue people. Each team was greatly interested in seeing who could save the most people in a given time.

This is discussed in detail as being somewhat typical of what may be done with many such number games.

Simon Says "Thumbs Up"

It is often desirable to have a single pupil recite all of a group of facts. Thus, a teacher may wish to have a pupil give several or all of the facts of addition or multiplication. But, unless she can in some way get the attention of the whole class and get all to attend silently to the combinations, the time of all but the one reciting is wasted. The following game shows how a resourceful teacher got the attention of the whole class in such a drill without any loss of time and without disturbance or extra material.

She called the game, "Simon Says 'Thumbs Up.'" One pupil stood before the class and showed the number cards. Another stood and gave the combinations. The class sat with "thumbs up." When a mistake was made, all thumbs were to turn down. The last to turn thumbs down stood to give the combinations, and the first to do so had the privilege of showing the cards.

Surprise Party

In using toy money and making change a good plan is to have a "Surprise Party." Ask the children how many would like to go to a surprise party. Of course, all want to go. Then ask what each would like to take and write his answer on the board, as apples, bananas, oranges, candy, nuts, etc. Next, appoint a clerk to step to the board and write the prices at which the articles can be bought at the store. Give the children toy money and let them buy the things of the clerk. Beads, blocks,

chalk, etc., may represent the articles. If the clerk makes a mistake, discharge him and let some one else take his place.

Playing Store

This game may be adapted to any of the primary grades and resembles somewhat the "Surprise Party." A table at the front of the room may serve as a counter of the store. Let the children determine the nature of the things to be bought and sold—fruit, flowers, toys, sporting goods, furniture, groceries, dolls, or candy. If varied frequently, the interest is keener. Things made in manual training or handwork periods may be used to form the stock. Charts may be used in place of actual objects. Added interest is gained by having toy money, toy scales, and a toy telephone. The children ask the prices of what they want, then determine how much they will buy. They should calculate the amount of their purchases so as to be able to detect errors on the part of the storekeeper.

Lunch Room

One child is the keeper of a make-believe lunch room. The other children pretend to buy their lunches from him. A chart showing the cost of each article of food should be posted.

Soldiers

The children form a company and choose their captain. The captain forms his men into ranks and files. He then drills them with number combinations. The answers should be given promptly, accurately, and with decision,

as becomes a soldier. Should they be given lazily or incorrectly, that soldier loses his rank, or may even be sent to barracks (out of the game). Sometimes two companies may combat with each other, the company giving most combinations correctly being victorious and taking the others captive or capturing their standard.

King of the Castle

A child, the king, sits on a chair in the front of the room. The other children try to dethrone the king by giving him number combinations which he cannot answer. As long as he gives correct answers, he remains on his throne in the castle; but, when he fails, the child who gave the combination becomes king in his place.

Tom Tiddler's Ground

A square on the floor represents "Tom Tiddler's Ground," in the center of which Tom is drawn. Some one asks, "Who will dare to go on Tom Tiddler's ground?" A boy in the class stands in front of the room to represent Tom. The child who dares to go on Tom Tiddler's ground is then questioned by Tom, who asks various number combinations. If correct answers are given three times, the child may stay on his ground without danger; but if he misses, he is chased out by Tom and some one else tries.

Horse Race

Lay before each child a flash card, as 6+4. Give a quick review to be sure each child knows his own card

correctly. Choose as many children for horses as there are aisles. The number cards are the hurdles, and the horses all start together. The horses giving correct combinations have surmounted the hurdles and pass on down the aisle. If a horse makes a mistake, the child holding that combination stretches his hand across the aisle. This barrier cannot be passed until the correct combination is given. The first horse to give all the combinations correctly in his aisle wins the race.

Elevators

Write a series of combinations as in the margin. To go up, add. To come down, subtract. Each combination represents a floor. If an incorrect answer is given, the elevator stops.

$$\begin{array}{r}4\\ \underline{2}\\ 3\\ \underline{2}\\ 5\\ \underline{2}\end{array}$$

Jack and the Bean Stalk

After telling the story say to the children, "Let us play that we are going to climb the bean-stalk ladder that Jack climbed, only we are going to climb it in a different way." Make a ladder of cardboard so that all the class can easily see it, or draw one on the board. In the spaces write number combinations in addition, subtraction, or multiplication. The one who can go up and down the ladder most rapidly (by giving the answers to the combinations) and without stumbling or slipping (making a mistake) can be Jack. The one who makes no mistakes and takes the shortest time to go up and down the

ladder makes the best Jack. A hard sum to overcome may be given for the giant and the one who can conquer this big giant is really "Jack-the-Giant-Killer."

Picking Apples

Draw an apple tree on the board with apples on it. Each apple has on its side a number combination. The children pick apples by giving the combinations. The one who picks the most apples wins the game.

Saving the Crew

A sinking ship is drawn on the board. Half of the deck of the ship is in the water. Combinations as $8+2$, 2×7, $8-2$, $10+2$, etc., are placed on the mast and on every part of the ship which is above water, to represent the crew of the ship waiting to be rescued. Then as many boats as there are pupils are drawn. Each child owns a boat and his initial is placed on it. Then, the one who saves the most of the crew — that is, tells the greatest number of combinations — is captain. For each man rescued, the child gets a flag which he places in his boat.

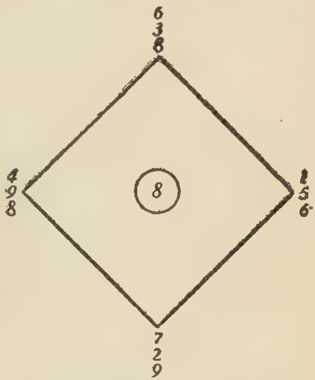
Playing Soldiers

Divide the class into two armies, A and B. Let them choose captains. The captain of army A asks a soldier of army B a combination. If the soldier cannot reply correctly, army B loses a soldier and army A gets him. If he replies correctly, however, he has the privilege of asking a soldier of army A a combination. The army

having the most soldiers at the end of a stated time wins the battle. This game may be used in subtraction, addition, multiplication, or division.

A Home-run

Divide the class into two baseball teams. Draw a baseball diamond upon the blackboard and write three or four numbers at each base, as in the figure. Some number to be added or to be used as a multiplier is written in the pitcher's place. The pupil makes a home-run by giving all the answers (sums or products) as the teacher, starting from the home plate, points to some number at each base. Thus, to use the game to multiply by 8, the teacher may point to 6, 9, 7, and 5, the pupil calling 48, 72, 56, 40, which constitutes a home-run. If a mistake is made, say at the second base, the umpire calls "out on second." The game is to see which team gets the most home-runs.



A Guessing Game

As a drill in addition, the teacher may say, "I am thinking of two numbers which added make 10." The pupils guess all of the combinations of numbers which added make 10, and when the last combination is given,

the teacher may call that the right one. The same drill may be used in multiplication. For example, the teacher may say, "I am thinking of two numbers whose product is 24." The pupils will guess two 12's, three 8's, six 4's. Or, the teacher may say, "I am thinking of the 'nine times' table." A child will ask, "Are you thinking of 81?" The teacher will say, "No, I am not thinking of 9×9." When the correct answer is given, the guesser may suggest the table and answer the questions, and thus it may be carried out between two children, the teacher acting as umpire.

Story Games

The stories with which the children are familiar, such as "Clytie," "Snowwhite," "Sleeping Beauty," etc., may be utilized in addition, subtraction, and multiplication drills. For example, in "Sleeping Beauty" a picture of the schoolhouse, thorns, huge walls, and a castle may be drawn on the board. Several combinations may be placed on each and a boy may start for the castle. If he succeeds in getting into the castle, having overcome all obstacles by giving the correct combinations on the various obstacles, he is dubbed knight and some emblem is pinned upon the lapel of his coat to signify as much. If he fails, some one else may try. Competition may be gained by having several such series on various boards, and several children trying to gain the castle first. The one who gives the combinations soonest is, of course, the honored one.

The children may take a ride with Clytie in her wonder-

ful seashell carriage, or visit the seven dwarfs in their mountains of gold. If so, the dwarfs or mermaids respectively may hold large cards on which are combinations, and these must be given before the child is allowed to arrive and visit the dwarfs' cave or Clytie's beautiful home. The children are enthusiastic over these simple devices.

These examples will serve to show the possibilities along this line of schoolroom games. A teacher interested in devising games to motivate her work will find suggestions on every hand.

GAMES OF MOTOR ACTIVITY

In schoolrooms devoted to a single grade, so that moving about does not disturb others, there are many games in which there is some movement; yet the amount of number work involved is great enough to meet the requirements of the classroom.

O–U–T Spells Out

The pupils may be arranged in a circle or remain in their seats. The first child gives some combination, as "5 and 7." The next in turn gives the sum and some other number, as "12 and 3." The next may say, "15 and 8." The game continues in this way, the first number given each time being the sum of the last two numbers given. For the first mistake a child makes he scores the letter O; for the second, U; and for the third, T. He is then out of the game.

Basketball

Large numbers are pinned on each child. The children stand in a circle about one child in the center who tosses a large ball to some one in the circle, at the same time calling out some number. If the child in the circle catches the ball and gives the correct sum, he exchanges places with the one in the center and is privileged to toss the ball. If he fails to catch the ball or gives the wrong sum, he remains in the circle.

A Column Relay Race

Divide the class into two teams. Give each a large card containing a number less than ten. The children stand in line. The leader of each team calls out his number and shows his card. The next in turn shows his number and adds it to the first, giving the sum. The next shows his number and adds it to the sum already given, and so on to the end of the line. The line completing the addition first wins the game if the sum is correct.

The numbers of one team should duplicate those of the other but should be arranged in different order. The sums will then be the same.

A Number Race

Divide the class into two teams — The Reds and The Blues. Write several numbers not larger than eighteen upon the blackboard, as 12, 17, 15, 13, and 18. Let the leader of each team go to the board. The remaining

pupils in turn, alternating from one side to the other, call some combination, the sum of which is on the board. Thus, a pupil from the Blues calls "7 and 8." The two pupils at the board see who can first cross out 15. After each has called a combination, the scores are taken. If the leader from the Reds has won 8 times and the one from the Blues has won 6 times, the score is 2 for the Reds. If a pupil calls a combination whose sum is not on the board, that takes one from the score of his side.

A Blackboard Relay Race

Before the class meets, the teacher places all the combinations to be drilled upon on the board, beginning at the ends of the board and working toward the center, at which a goal is drawn. The same combinations are on either side of the goal but written in a different order. The class is divided into two teams. At a given signal, the leader of each team runs to the board and records the result of the first combination, returns and gives his chalk to the next in line, who records the next result, giving his chalk to the next in order, and so on until the goal is reached.

Old Witch

Have an old witch, mother, and children. Give the children numbers such as 15, 14, 16, etc. The old witch comes to the door and says, "I want a child." "Is it 8 and 7?" asks the mother. "No, it is not 15," says the witch. "Is it 7 and 7?" asks the mother. "Yes, it is 14," says the witch. "You must catch me then," says

14, and he runs. If the old witch catches him, he is hers; if not, he is free. If the child does not know the combination and is not ready to run, he belongs to the old witch. The teacher may take the place of the mother, so that new combinations may be given. Addition, subtraction, and multiplication combinations may be used.

Shepherd and Wolf

One child is the shepherd and one is the wolf, and all the rest are sheep. The shepherd stands at one end of the room, the sheep at the other, and the wolf between the two. The shepherd has some number, as 5. The sheep each have numbers as 1, 2, 3, 4, 5, etc. The wolf gives a number, as 12. When the wolf says 12, the number that, added to the shepherd's number 5, will make 12 — *i.e.* 7 — must run across the room. If the wolf says 8, then number 3 runs, etc. If a sheep fails to run when he should, he belongs to the wolf, who takes him to his den. If he runs and is not caught, he may go back to his former place.

Beast, Bird, and Fishes

Children stand in the circle with one child in the center. The one in the center points quickly to a child in the circle, giving him a combination to answer. Then the child in the center counts 1, 2, 3, 4, 5. The one pointed to must give the answer before 5 is counted or he is "out" of the game.

Who Taps

One child is outside of a circle of children. He taps a child on the back. "Who taps?" demands the one tapped. The answer given is a combination. Then the other must reply with the answer to the combination or else go outside of the circle.

Catch the Ball

The children form two lines. Each has a number pinned on in full view. Some one tosses a ball, or a bean bag, to some one on the other side, who gives the sum (product or difference) of his number and that of the pitcher. If correct, he tosses it back to some one on the first side, who answers in the same way. When a mistake is made, the one making it drops out of the game and the ball or bag goes to the head of the opposite side to toss.

Number Basket Upset

The children sit in a circle with numbers pinned on them. Some one stands in the center and gives a combination three times in succession, as "7 and 2, 7 and 2, 7 and 2." The child with 9 must say "9" before the one in the center has finished. If he is not quick enough, he forfeits his seat to the child in the center and then gives the combinations. The one in the center may at any time say, "Number basket upset," and every child must change his seat, the one in the center also trying to get a seat. There will be one child left without a chair who must be in the center and give the combinations.

Grunt

Each child is given a number, and all but one form a circle. One child in the center is blindfolded. He has a pointer or small rod. The children march around him in a circle. When the one in the center taps the floor with the rod, all stop. He touches some one in the circle, who takes one end of the rod and gives the sum of his number and the number of the one who is blindfolded. The one blindfolded then "guesses" the number of the one touched, which he does, of course, by subtracting his number from the sum given. If correct, the one touched is blindfolded.

A Jumping Game

Two children "turn" a rope and the others stand in line ready to jump the rope in turn. Each child is to jump as many times as he can, counting by some number, as 5, each time he jumps. Whoever counts farthest wins. If he makes a mistake in counting, he stops jumping. This gives practice in adding equal addends.

Ball and Hoop

A hoop is suspended in the room. The object of the game is to throw a ball through the hoop. The players are divided into sides and take turns alternately. Each time a child throws the ball through the ring, the captain of the opposing team challenges him with some number combination. If he can answer it, he scores two for his

team; but, if he cannot answer it correctly, he loses one point for his team.

Hot Butter Beans

Instead of hunting for a bean, as in the real game, let the children hunt for a numbered block. Let one child hide the block. When ready, the others hunt. The hider says "hot" or "cold" according to the nearness of the children to the block. The child who finds the block adds the number to his score if he can call the combination quickly. He then hides a block. The number on the block is changed each time. Let the final score be 100, and the child which gets it first wins the game.

Fencing

The children stand in two lines facing each other. Each child must be opposite an opponent. Then some number is decided upon, as twelve. One child gives a number, such as seven, and his opponent is required to give the number which must be added to make twelve. This is done all along the line. Whenever a child misses, the mistake is recorded against him. The mistakes are counted after a given time, and the side which has made the fewest mistakes wins.

OUT-OF-SCHOOL GAMES

Most of the games played by children out of school may be adapted to number games without taking away the recreative elements of the games. These games

should not be given as a requirement, nor should they involve enough number work to task the children's abilities. The purpose of them is not drill in number work, but they are given to motivate the classroom work by showing a use of the things they are learning. The following will suggest ways of using the well-known games.

Fox and Goose

Large numbers are pinned on each child. Choose one child for the fox and one for a goose. The other children stand in pairs about the playground with linked arms. The fox tries to catch the goose. In order to escape from the fox, the goose may link arms with any one of any couple, first calling out the sum of the numbers of the couple. The pupil with whom the goose did not link arms then becomes goose. If the goose calls the wrong sum, she is not safe and must run to another couple and try again, or be caught.

Pussy Wants a Corner

All children are numbered with large visible numbers, each two children having the same number, and all but one has a corner (a base of some sort). The child without a corner goes from one to another saying, "Pussy wants a corner." The one to whom it is said asks, "What corner?" The reply is some two numbers whose sum (or product) is represented by the children. The two children having the sum (or product) exchange places. The child without a corner tries to get one of the vacant places. Thus, if he answers "7 and 8," the two 15's

exchange places. If he can get one of the places, the one left becomes "pussy."

Squat Tag

Pin a visible number on each child. Let one child be chosen as "it." He chases some one of the other children. To prevent being caught, the one who is chased squats down, giving the sum (or product) of the number pinned on "it" and his own number. If the sum (or product) is incorrect, he may be caught and then becomes "it."

Fox and Geese

Divide the class into foxes and geese. Give the foxes a certain amount of space and geese the same amount. Geese fly over into the territory occupied by the foxes. If a goose is caught there, she must answer a combination. If she answers correctly, she may go free; if not, she must be a fox. Foxes may visit the home of geese and take away with them all the geese which cannot answer correctly the combinations put to them. The aim of the geese is not to be caught; or, if caught, to answer correctly the combination given so that they may go free.

Tap the Rabbit

All the children but one, who is to be the leader, form in a ring. Each child has a number pinned upon him. The leader runs around the ring and taps some one in the ring and calls a number. The one tapped gives the combination of his number with the number called, before he starts around the ring in the opposite direction. The

one reaching the vacant place first stays in the ring; the other is leader. This game may be played with addition, multiplication, or subtraction combinations.

Three Deep

Have two rings of children, one child in front of the other. Have two children outside of the ring, one pursued by the other. All have numbers pinned on front and back of them. The one pursued runs and stands in front of a child in the ring, who quickly gives the sum of his number and the number of the child who stood in front of him and immediately chases the former pursuer. If a child is caught, he must become the pursuer.

Drop the Handkerchief

The children form a ring. A number is placed in the center of the ring. One child runs around the outside of the ring with a handkerchief. He throws it behind some one in the ring and calls some number less than the one in the ring. The child behind whom it is thrown quickly subtracts the number called from the number in the ring, and starts around the ring in the same direction. If he overtakes the one who dropped the handkerchief before he reaches the vacant place, he then drops the handkerchief. If he does not, he is out of the game.

Hands Up

The children form a ring around the room or on the playground. A number is pinned on each child. A catcher moves around inside of the ring. The children

tantalizingly hold up both hands above their heads, tempting the catcher to tap them while the hands are raised. If he can succeed in touching any of them while their hands are raised, and at the same time can give the combination of his number and that of the one caught, he takes the place of the one caught, who then becomes catcher.

Hop and Shake Hands

A ring is formed and the children have numbers pinned on them. One person is "it." He runs around the ring, taps some one on the back, and calls some number. Before the one tapped can run, he must add his number to the number called and give the sum. After he gives the sum, he runs in the opposite direction to meet the tapper. When they meet, they shake hands and then, hopping on one foot, they try to see which one can get in the vacant place first.

Witch's Circle

One child is the witch. All the children run across the line and, if the witch catches a child, she places him in a small circle containing a number combination. If he can answer the combination in the small circle as soon as he is placed in it, he may go free; if not, he must stay until the old witch releases him.

In the Pool

There is one child at each corner of a square, making in all four players called "strikers." The other players

stand in the middle of the square called "the pool." The players in the pool have numbers pinned on them. This game may be used in multiplication or addition. Let us suppose that it is the 4's in multiplication. The four players at the corners throw a ball from one to another. When they catch the ball, they try to touch one of the children in the pool with it. If that child's number is 6 and he at once calls "24," he may stay in the pool. If he makes a mistake, it is the "striker's" place to correct him. If the "striker" does not do this, some one in the pool may call out the answer. The child who calls the answer may then take the "striker's" place. The object of the game is to see who can stay "striker" the longest.

Have You Seen My Sheep

The children form a circle, one child outside being the shepherd. The shepherd taps some one on the back, saying, "Have you seen my sheep?" The one tapped will reply, "No, how many pounds did he weigh?" The shepherd will reply with some such combination as 3×5, or $8+9$, or $10-4$, and then start to run around the circle. The one tapped must give the answer to the combination before he can run after the shepherd, who tries to get around to the vacant place before he is caught. If the shepherd is not caught, the one tapped becomes shepherd. If the one tapped cannot answer his combination correctly, he must take his place in the center of the circle.

CHAPTER IX

COMMON FRACTIONS

SINCE a little objective work in fractions is done in the lower grades, before a systematic study of the subject is taken up in the fifth and sixth grades, it seems best to break up the discussion into two parts, the first relating to the work of the primary grades, and the second to the work of the intermediate grades.

I. EARLY WORK IN FRACTIONS

The work done in fractions in the first four grades is preparatory to the written work with symbols in the fifth grade. A clear notion of fractions cannot be obtained from the symbols. The first work must be objective. By the use of objects, diagrams, etc., the child sees a half, a fourth, or any fractional unit as an individual unit, just as a quart, or a foot. He adds, subtracts, multiplies, or divides a number of these units just as he would a number of integral units. Let him get the names of these fractional units by having him notice the similarity of sound between the number of equal divisions and the names of the parts. Thus, one of *four* equal parts is a *fourth;* one of *five* equal parts, a *fifth;* one of *six*, a *sixth;* etc.

First study the fractions in related groups as halves and fourths, then halves, fourths, and eighths, etc., and learn the relations among them. This should be done objectively so that a child sees that a half equals two fourths, just as he sees that a quart equals two pints, etc.

If the work is objectively presented, the child is able to use intelligently the common fractions before knowing the notation for them. Thus, he knows that a pint is half of a quart, that a quart is a fourth of a gallon, etc. When a fractional unit is studied, it should not only be introduced by some real need of such a relation, but followed by real applications. If halves and fourths are studied, there are many real applications. Thus, if milk is 10 cents per quart, how much is a pint worth? If cider is 20 cents per gallon, how much is a quart worth? If walnuts are 80 cents a bushel, how much is a peck worth? In this way, through handling objects, folding paper, making diagrams, etc., the pupil comes to think of a fractional unit as a concrete thing, just as an integral unit. Thus he gets the first of the three steps in the proper teaching of fractions, viz.: (1) the meaning, (2) the notation, and (3) the manipulation.

The Notation of a Fraction

It is very important that the pupil should get the full meaning of the notation of a fraction, for all the manipulations depend upon the notation. It is not so important for him to get the names "numerator" and "denominator" when the notation is first presented; in fact, it may be

better to avoid using these terms until after the function of the terms is well understood.

In developing the notation, take some object or diagram, and divide it into a number of equal parts, say 4. Then have the children point out some number of these fourths, as 3. Now write 3 on the board and ask if that alone tells us that there are three fourths. Show 3 of some other unit as dollars, cents, feet, etc., and let the pupils see that the 3 shows "how many" and that some other sign is needed to show what they are, as \$3, 3¢, 3 ft., etc. Then show that a sign is needed to show what the 3 fourths are, and that the sign is a 4 written below the 3. Thus in $\frac{3}{4}$ the 3 shows "how many" and the 4 shows what they are.

To test the pupils' understanding of the notation, write several fractions and ask of each fraction, "How many things?" "What are they?" as:

$$\frac{3}{4}, \frac{5}{6}, \frac{2}{3}, \frac{7}{8}, \frac{5}{7}, \frac{3}{5}, \frac{7}{9}, \frac{3}{10}, \text{etc.}$$

This notation is the basis of all the processes and must be thoroughly understood. But when it is understood, the manipulations follow from those of integers.

The Fundamental Processes in Fractions

In teaching the processes they should be related to those already known. Thus, just as $\$3+\$1=\$4$, so $\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$. Just as 3 ft.+8 in. cannot be added until changed to a "like unit," so $\frac{3}{4}+\frac{1}{8}$ cannot be added until changed to a like unit. In the same way, subtraction follows that of integers.

Just as $3\times\$5$ means $\$5+\$5+\$5$, or \$15, so $3\times\frac{5}{8}=\frac{5}{8}+\frac{5}{8}+\frac{5}{8}=\frac{15}{8}$; or 3 times 5 of any unit is 15 of that unit.

When we speak of multiplying by a fraction, we are giving a new meaning to multiplication. The process is based upon partition, not upon addition, as was the case when multiplying by an integer. But just as $\frac{3}{4}$ of an object is found by first dividing the object into 4 equal parts, then taking 3 of them; or $\frac{3}{4}$ of a group, as 16, is found by dividing 16 into 4 equal groups, then taking 3 of the groups; so to find $\frac{3}{4}$ of a fraction means the same thing. Thus, $\frac{3}{4}$ of $\frac{3}{5}$ means that we first divide each of the three fifths into 4 equal parts, making three twentieths, then take 3 of them, making $\frac{9}{20}$. That is, $\frac{3}{4}$ of $\frac{3}{5}=\frac{9}{20}$.

Also, just as $\$8\div\$2=4$ means that \$8 will contain \$2 four times, so $\frac{8}{9}\div\frac{2}{9}=4$ means that $\frac{8}{9}$ will contain $\frac{2}{9}$ four times. Likewise, 8 of any unit will contain 2 of that unit four times.

And, just as one cannot tell how many times 2 ft. will contain 8 in. without first knowing how many inches in 2 ft., so one cannot tell how many times one fraction will contain another until they are both expressed in the same unit. Thus, the first work in the division of fractions is based upon the measurement phase of division of integers.

II. FRACTIONS COMPLETED

Objective Presentation of Fractions

It was shown in the first part of this chapter that fractions are studied objectively in the lower grades and that pupils gain naturally and informally one of the several

ideas involved in fractions. This is, that *a fraction is one or more of the equal parts into which some whole has been divided.* This notion is the basis upon which the fundamental processes are built.

In the lower grades and through the use of objects, the child also begins to get one of the real uses of fractions, viz., its use as a means of expressing a relation. He knows that, if he has 4 marbles and loses one half of them, he will have but 2 left; and also that, if he has 4 marbles and loses 2 of them, he has lost half of them. This use of a fraction to express a relation comes slowly and gradually to him through such related questions as, "If you have 6¢ and spend half of them, how much will you spend?" and, "If you have 6¢ and spend 3¢, what part of your money do you spend?"

The Ideas Involved in a Fraction

There are three ideas involved in a fraction: (1) a fraction is one or more of the equal parts of a whole; (2) a fraction is an indicated division; and (3) a fraction is an expression of the *ratio* of one quantity to another. The development of the fundamental processes from the corresponding processes with whole numbers depends upon the first of these ideas. Much of the pupil's later work depends upon the second and third ideas, and these must be firmly fixed in his mind before the completion of the subject; but they need not be taken up until after the development of the fundamental processes with fractions — addition, subtraction, multiplication, and division.

A Fraction as a Part of a Whole

First through dividing objects, folding paper, using diagrams upon the blackboard, or with similar illustrations, get the pupil to see the principle involved in naming the fractional unit. That is, have him see that, if a thing is divided into *four* equal parts, each part is a *fourth* of the whole; if in *five* equal parts, each is a *fifth;* if in *six*, each is a *sixth;* etc. Next have him count the fourths, fifths, sixths, etc., just as he would count any unit, and thus $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, etc., will become just as real numbers of things as 3 feet, 4 quarts, and 5 bushels are.

It is now important that the pupil get the correct idea of the notation from this point of view of a fraction as a part of a whole. When he points out a number of fractional units as 3 or 4 of them, have him see that just as in three dollars or three feet, 3 only tells *how many*, and that, to express the whole idea, the sign of the unit is necessary as $3 and 3 ft.; so in fractions three fourths is written $\frac{3}{4}$, the 3 denoting the number of things and the 4 telling what they are, just as "$" or "ft." does.

A little drill upon such fractions as $\frac{4}{5}$, $\frac{5}{6}$, $\frac{3}{8}$, $\frac{5}{9}$, $\frac{7}{10}$, etc., asking, "How many?" pointing to the numerator; then, "What are they?" pointing to the denominator, will soon fix in mind the function of the two terms of a fraction, a thing much more important than to know their names.

Knowing the meaning of the terms, however, the names may be so given as to make it easy to remember them. Thus, writing some fractions on the blackboard, the

pupil will tell you that the upper term tells "how many" or the *number* of things. It is then the *numberer* or *numerator*, and through the similarity in sound he fixes the name and its meaning. He then tells you the lower term tells what the number of things denoted by the upper term are. Get him to see then that it names the unit or is the *namer* and hence is called the *denominator*. The pupil can remember this if the teacher will call his attention to the fact that when we *name* a man to run for office we *nominate* him, and that we are "the nominators"; and likewise the term that *names* the fraction is the *de-nominator*.

A Fraction as an Indicated Division

It is essential in changing common fractions to decimals, and in other work that follows, that a child get a fixed impression of a fraction as an indicated division; but it is hard to give him a satisfying basis for this notion. It is as well to make no attempt to give such an idea until after the processes have been mastered. But, having mastered the processes, he knows that finding $\frac{1}{4}$ of anything means to divide it into 4 parts. So $\frac{1}{4}$ of 8 means $8 \div 4$. He also knows that $\frac{1}{4}$ of 1 is $\frac{1}{4}$, and that $\frac{1}{4}$ of 8 is eight times as large, and hence is $\frac{8}{4}$. From this and similar examples get him to see the truth of the following statements:

$$\tfrac{1}{4} \text{ of } 8 = \tfrac{8}{4}, \text{ and } = 8 \div 4;$$
$$\tfrac{1}{3} \text{ of } 6 = \tfrac{6}{3}, \text{ and } = 6 \div 3;$$
$$\tfrac{1}{5} \text{ of } 15 = \tfrac{15}{5}, \text{ and } = 15 \div 5.$$

Hence,

$$\tfrac{8}{4} = 8 \div 4;$$
$$\tfrac{6}{3} = 6 \div 3;$$
$$\tfrac{15}{5} = 15 \div 5.$$

Follow this by having the pupils express such forms as $3 \div 4$, $5 \div 7$, $3 \div 8$, $2 \div 5$, etc., as fractions.

A Fraction as a Ratio

This conception is a slow growth that comes from using a fraction to express the relations of simple objects from the very first conception of a fraction. The pupil may get much help, however, from the meaning of the notation of a fraction. Thus, $\frac{5}{8}$ is 5 of the 8 equal parts; that is, it is 5 of 8. So, if the pupil has eaten 5 of 8 apples, he has eaten "5 of 8" or $\frac{5}{8}$ of them. If he has 11 marbles and loses 8, he has left 3 of 11 or $\frac{3}{11}$. And thus he comes gradually to see that the ratio of one number to another is the fraction whose numerator is the first number and whose denominator is the second.

Objectively the pupil may see that the ratio idea is consistent with his first idea of a fraction as one or more of the equal parts of a whole.

$$\begin{array}{ccccccccc} \multicolumn{7}{c}{\overbrace{\hphantom{\bullet\quad\bullet\quad\bullet\quad\bullet\quad\bullet\quad\bullet\quad\bullet\quad\bullet}}^{\frac{7}{9}}} & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{array}$$

Thus, if we consider 9 things, say, as the whole, then each is one of the nine equal parts that made the whole, or $\frac{1}{9}$ of the whole. And 7 of them is $\frac{7}{9}$ of the whole, and thus the ratio of 7 to 9 is $\frac{7}{9}$; that is, 7 is $\frac{7}{9}$ of 9.

ADDITION OF FRACTIONS

After the nature and notation of a fraction is understood, the addition of fractions presents nothing new; that is, the work has the same meaning and follows the same fundamental law, that only like things can be added, which the pupil has had in whole numbers. The important thing is that the work be carefully graded so as to introduce but one form at a time and that the pupil sees that it is like the addition he already knows. The gradation should be: (1) two fractions whose units are alike; (2) two fractions whose units are unlike, but where one can be changed to the other; (3) two fractions whose units are unlike and a new unit must be found to which both can be changed.

Adding Fractions with Like Units

The pupil knows that the upper term, or numerator, shows the number of things considered and that the lower term, or denominator, shows what they are. So just as $2+$3=$5, or 2 ft.+3 ft.=5 ft., so 2 of any unit +3 of a like unit =5 of that unit. Hence, $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$, or $\frac{2}{9}+\frac{3}{9}=\frac{5}{9}$, etc. And thus he should see clearly that when the units are alike, the numerators only are added for they are the number of things that are combined. The denominator merely shows what they are as do the abbreviations and signs in denominate numbers.

If the work is presented in this way, pupils will never attempt to add two fractions by adding numerators and denominators, a mistake often seen in grammar and high schools.

ADDING FRACTIONS WITH RELATED DENOMINATORS

The pupil knows from his work in whole numbers and with denominate whole numbers that 2 bu.+3 pk. does not equal 5 bu. or 5 pk.; that, before he can combine them into one number, they must both be expressed in like units, and that he must express them as 8 pk.+3 pk. before he can add them. So in $\frac{1}{4}+\frac{3}{8}$, he knows that, although he has 1 thing + 3 things, he cannot call them 4 of either, for 1 of them is a *fourth* while the other 3 are *eighths*. So, before they can be added, the $\frac{1}{4}$ must be expressed as $\frac{2}{8}$. Then $\frac{2}{8}+\frac{3}{8}=\frac{5}{8}$.

It will be noticed that this class of addition calls for a knowledge of the relations of certain related fractional units. While a study of the idea of a fraction as one or more of the equal parts of a whole through the use of objects has no doubt shown the pupil certain relations, this class of addition furnishes a motive for further comparison, and the pupil should now become as familiar with the relations of those units likely to occur in life as he is with the relation of the units in the various tables of denominate numbers. This he does through the use of objects and diagrams.

Thus, a diagram like the following shows him the relation of halves, fourths, and eighths.

<table>
<tr><td colspan="4">1/2</td><td colspan="4">1/2</td></tr>
<tr><td colspan="2">1/4</td><td colspan="2">1/4</td><td colspan="2">1/4</td><td colspan="2">1/4</td></tr>
<tr><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td></tr>
</table>

The following shows the relation of thirds, sixths, and ninths.

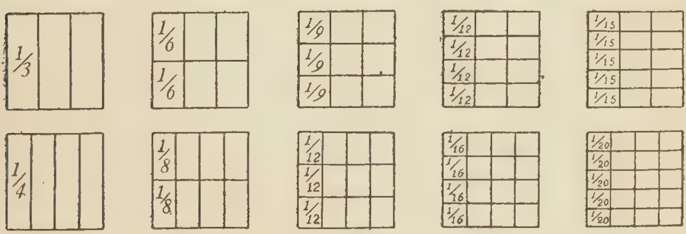
<table>
<tr><td colspan="6">1/3</td><td colspan="6">1/3</td><td colspan="6">1/3</td></tr>
<tr><td colspan="3">1/6</td><td colspan="3">1/6</td><td colspan="3">1/6</td><td colspan="3">1/6</td><td colspan="3">1/6</td><td colspan="3">1/6</td></tr>
<tr><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td><td colspan="2">1/9</td></tr>
</table>

Adding Fractions with Unrelated Units

The adding of such fractions as $\frac{2}{3}$ and $\frac{3}{4}$ brings up the auxiliary process of changing fractions to like units or to "a common denominator." Such a problem as this, then, should be presented in order to furnish an aim or motive for such a process.

The Least Common Denominator

Using the problem presented; namely, $\frac{2}{3}+\frac{3}{4}$, let us find a unit to which both 3ds and 4ths can be changed.



By diagrams show that if 3ds are divided into 2 equal parts we have 6ths; if into 3, we have 9ths; if into 4, we have 12ths; etc. Have the pupil see that the denominators of the fractions to which 3ds can be changed are the successive *multiples* of 3 and thus introduce the term

multiple. If that is always so, then 4ths can be changed to 8ths, 12ths, 16ths, 20ths, etc. Let him see from the diagram that that is just what he would have.

He is now led to see that the denominators to which 3ds and 4ths can be changed will contain both 3 and 4. He should now be able to find by inspection the unit to which two or more fractional units can be changed.

Reducing Fractions to New Units

Using the diagrams above, lead the pupil to see that when 3ds are changed to 6ths, while the units are half as large, there are twice as many of them; changed to 9ths, they are one third as large but three times as many; etc. Thus, $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}$, etc., which may be found mechanically by multiplying both terms by the same number.

Here it is well to call attention to the use he will have of the truth illustrated here — that both terms of a fraction may be multiplied or divided by the same number without changing the value represented by the fraction.

Now, to change $\frac{2}{3}$ and $\frac{3}{4}$ to like units, ask, "What denominator will contain both 3 and 4?" "By what must 3 be multiplied to give 12?" "Then by what must each term of $\frac{2}{3}$ be multiplied to change it to 12ths?" and then ask similar questions about $\frac{3}{4}$.

Reducing Improper Fractions

When adding the fractions proposed above, we have $\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}$. Now the pupil will observe that $\frac{17}{12}$ is not a fraction in the sense that he has thought of a

fraction, for in dividing a whole into 12 parts to get 12ths there are but 12 of them in any whole. So this may be called an *improper fraction,* since 12 of the 17 twelfths will make a whole and 5 of them will remain. Hence, $\frac{17}{12} = 1\frac{5}{12}$. Unless the pupil has come to the second notion of a fraction, that it is an expressed division, he sees no meaning in "dividing the numerator by the denominator" as a method of reduction. The natural development from his first idea of a fraction is that, since the denominator shows the number of parts into which the whole was divided, it is the number of parts that it takes to make a whole. Thus, in $\frac{25}{8}$, it takes 8 of the 8ths to make a whole; so there will be as many wholes as there are 8's in 25, or 3 of them and 1 remaining. So $\frac{25}{8} = 3\frac{1}{8}$. Through such a thought as this, he sees that mechanically the answer is obtained by "dividing the numerator by the denominator," the rule usually given.

Forms of Adding Mixed Numbers

There are several forms of adding mixed numbers. The two most common are given below:

Form A	*Form B*	
		24
$13\frac{1}{8} = 13\frac{3}{24}$	$13\frac{1}{8}$	3
$16\frac{3}{4} = 16\frac{18}{24}$	$16\frac{3}{4}$	18
$17\frac{5}{6} = 17\frac{20}{24}$	$17\frac{5}{6}$	20
$46\frac{41}{24} = 47\frac{17}{24}$	$47\frac{17}{24}$	41

Form A is an uneconomical procedure for it requires rewriting the whole numbers and the common denominator

with each fraction. In form B the common denominator is written above and the numerators are written in a form easily added.

Adding Special Fractions

The pupils should always be alert for special combinations that will save work. There are two types of exercises in the addition of fractions that should be noticed. These two types are shown by the following examples:

First type: $\frac{3}{4}+\frac{5}{8}+\frac{1}{2}+\frac{3}{8}+\frac{1}{4}$.

Instead of changing all to 8ths, the pupil should collect those whose units are alike. So $\frac{3}{4}+\frac{1}{4}=1$; $\frac{5}{8}+\frac{3}{8}=1$; and, hence, the sum $=2\frac{1}{2}$.

Second type: $\frac{1}{4}+\frac{1}{5}$.

Since 4 and 5 have no common factor and since the numerators are each 1, the *new* numerators will be 5 and 4 respectively. Hence, the sum is $\frac{9}{20}$ which is "the sum of the denominators over their product."

SUBTRACTION OF FRACTIONS

As in addition of fractions, the pupil should see that the meaning and the process of subtraction are the same as in whole numbers. The gradation is not so important as in addition, for all the principles needed have already been taught. Yet, the grading is exactly the same as in addition except that the passing from one type to another may be done more quickly. The grading is:

(*a*) Like units, as $\frac{5}{8}-\frac{3}{8}=\frac{2}{8}=\frac{1}{4}$;

(*b*) Related units, as $\frac{7}{8}-\frac{1}{4}=\frac{7}{8}-\frac{2}{8}=\frac{5}{8}$;

(*c*) Unrelated units, as $\frac{2}{3}-\frac{1}{4}=\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$.

Forms of Work in Mixed Numbers

When the fraction in the minuend is greater than the one in the subtrahend, they are subtracted as in pure fractions. With pupils in the lower grades the work may be written down as in form B under addition. With mature pupils the change to like units and the subtraction may be done mentally and the numerators of the like fractions not written. That is,

$$\begin{array}{l} 46\frac{2}{3} \\ \underline{13\frac{1}{4}} \\ 33\frac{5}{12} \end{array} \qquad \text{not} \qquad \begin{array}{l|r} & \underline{12} \\ 46\frac{2}{3} & 8 \\ 13\frac{1}{4} & 3 \\ \hline 33\frac{5}{12} & 5 \end{array}$$

When the fraction in the minuend is smaller than the one in the subtrahend, work is saved if the fraction in the subtrahend is subtracted from 1 and the result added to the fraction of the minuend. Thus, in

$$\begin{array}{l} 46\frac{1}{3} \\ \underline{15\frac{3}{4}} \end{array} \qquad \text{we think} \qquad \begin{array}{l} 45+1\frac{1}{3} \\ \underline{15+\ \frac{3}{4}} \end{array}$$

Now we may use $1-\frac{3}{4}+\frac{1}{3}=\frac{1}{4}+\frac{1}{3}=\frac{7}{12}$, instead of $1\frac{1}{3}-\frac{3}{4}=\frac{4}{3}-\frac{3}{4}=\frac{16}{12}-\frac{9}{12}=\frac{7}{12}$.

Subtracting Special Fractions

When the numerators are each 1 and the denominators have no common factor, the result may be written down at once. Thus $\frac{1}{2}-\frac{1}{5}=\frac{3}{10}$, for the *new* numerators are 5 and 2 respectively and the common denominator is 10; that is, the result is "the difference of the denominators over their product."

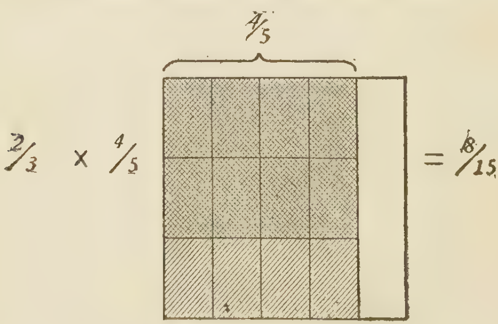
MULTIPLICATION OF FRACTIONS

Multiplying a Fraction by a Whole Number

The pupil should be shown clearly that multiplying by a whole number and multiplying by a fraction have very different meanings. He has learned that multiplying by a whole number is a means of saving addition when the addends are alike. So, just as $3 \times \$2 = \$6$, 3×2 ft. = 6 ft., and 3×2 of any unit $= 6$ of that unit, so $3 \times \frac{2}{7} = \frac{6}{7}$, $3 \times \frac{2}{5} = \frac{6}{5}$, etc.; that is, the numerator being the number of units under consideration is the number that is multiplied. The denominator is merely written to show what the units are, as were the "\$" and "ft." in the examples shown above.

Multiplying a Fraction by a Fraction

To find the product of $\frac{2}{3} \times \frac{4}{5}$ means to find $\frac{2}{3}$ of $\frac{4}{5}$. But to find $\frac{2}{3}$ of anything always means the same. It means



to divide the thing into 3 equal parts and then take 2 of them. Now, 5ths divided into 3 equal parts gives 15ths,

so $\frac{1}{3}$ of $\frac{4}{5}=\frac{4}{15}$. Hence, $\frac{2}{3}$ of $\frac{4}{5}=2\times\frac{4}{15}=\frac{8}{15}$. Hence, it is seen that the product of two fractions is found by taking the product of the numerators for the numerator of the product, and the product of the denominators for the denominator of the product. The above product may be shown objectively.

Multiplying by a Mixed Number

To multiply by a mixed number will involve multiplying a whole number by a fraction. As seen above, to multiply anything by a fraction requires both a division and a multiplication. But, just as $4\div2\times3$ gives the same result as $4\times3\div2$ (that is, the order may be changed), the multiplication by the numerator may take place before the division. Thus, $13\frac{3}{4}\times25$ is found as in the margin.

$$\begin{array}{r} 25 \\ 13\tfrac{3}{4} \\ 4\overline{)75} \\ \hline 18\tfrac{3}{4} \\ 75 \\ 25 \\ \hline 343\tfrac{3}{4} \end{array}$$

Enough drill should be given to fix the habit of putting the partial products in the right place. Some pupils will put the 5 of the second product under the 1 of the first.

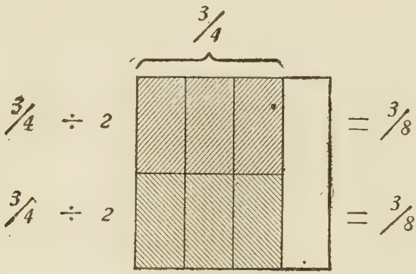
Where both multiplier and multiplicand contain fractions and the whole numbers are small, work may be saved by changing both to improper fractions. Thus, $3\frac{1}{2}\times12\frac{5}{8}=\frac{7}{2}\times\frac{101}{8}=\frac{707}{16}=44\frac{3}{16}$.

DIVISION OF FRACTIONS

The pupil who has understood the two meanings of division of whole numbers will have no difficulty in seeing the corresponding meanings when applied to fractions, which to him are now a sort of denominate number.

Dividing a Fraction by a Whole Number

Any division by an abstract whole number may be interpreted as the *partition idea of division;* that is, as a division into parts. Just as $\$8 \div 2 = \$4$, 8 ft. $\div 2 = 4$ ft., and 8 of any unit $\div 2 = 4$ of that unit, so $\frac{8}{9} \div 2 = \frac{4}{9}$, $\frac{8}{5} \div 2 = \frac{4}{5}$, etc.



Likewise, from the same meaning of dividing into parts, each fractional unit may be divided into parts as shown above. So, when the numerator cannot be divided without a remainder, the denominator may be multiplied by the divisor.

This last fact is also shown from the same meaning; that is, to divide by 2 is to find half of; by 3, to find a third of; by 4, to find a fourth of; etc. Thus, $\frac{3}{4} \div 5 = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20}$.

Dividing a Fraction by a Fraction

To divide by a fraction cannot mean partition, for a thing cannot be divided in $\frac{3}{4}$ equal parts. Such an expression has no meaning. A thing can only be divided

into 2 or 3 or 4 or some whole number of equal parts. Hence, this type of division is the *measuring idea of division;* that is, it is finding how many times the dividend will contain the divisor. Just as 8 qt. $\div$ 2 qt. $=4$, or 8 ft. $\div$ 2 ft. $=4$, so 8 of any unit will contain 2 of that unit 4 times. Then, $\frac{8}{9}\div\frac{2}{9}=4$, $\frac{8}{5}\div\frac{2}{5}=4$, etc.; that is, just as in any denominate number, the units must be alike, then the numbers of them are the numbers used in the actual division.

$$\frac{2}{3}\div\frac{3}{4}=\frac{8}{12}\div\frac{9}{12}=8\div 9=\frac{8}{9}$$
$$\frac{2}{5}\div\frac{5}{6}=\frac{12}{30}\div\frac{25}{30}=12\div 25=\frac{12}{25}$$

Dividing by Inverting the Divisor

There are several ways of developing the inverted divisor method of division. Perhaps the simplest method is to analyze the work done in changing to like units. Thus, take the example given above of $\frac{2}{3}\div\frac{3}{4}$. To get like units, both terms of the dividend were multiplied by the denominator of the divisor. Then both terms of the divisor were multiplied by the denominator of the dividend. But when thus reduced, only the new numerators, $2\times 4\div 3\times 3$, were actually used in the division. This quotient expressed as a fraction is $\frac{2\times 4}{3\times 3}$, which is the product of $\frac{2}{3}\times\frac{4}{3}$. But this is the product of the dividend by the divisor inverted.

Another method in quite common use is to develop the process from the two known facts that multiplying the divisor divides the quotient, and dividing the divisor

multiplies the quotient; and that with the divisor remaining constant, multiplying or dividing the dividend multiplies or divides the quotient by the same number. With these two principles in mind, the development is as follows:

$$1 \div 1 = 1;$$
$$1 \div \tfrac{1}{4} = 4 \times 1 = 4;$$
$$1 \div \tfrac{3}{4} = \tfrac{1}{3} \times 4 = \tfrac{4}{3}.$$

From several examples like this, the pupil gets the following generalization, *that the quotient of one divided by any fraction is the fraction inverted.* The second step is as follows:

$$\text{Since}\quad 1 \div \tfrac{3}{4} = \tfrac{4}{3},$$
$$\text{Then}\quad \tfrac{2}{3} \div \tfrac{3}{4} = \tfrac{2}{3} \times \tfrac{4}{3} = \tfrac{8}{9}.$$

That is, to divide by a fraction, invert the divisor and multiply.

Dividing a Mixed Number by a Whole Number

While no new principles are involved, the method of dividing a mixed number by a whole number needs to be taken up. Thus, to divide $3896\frac{2}{3}$ by 7, we have

$$\begin{array}{r} 7\underline{)3896\frac{2}{3}} \\ 556,\ 4\frac{2}{3}\ \text{rem.} \end{array}$$

$4\frac{2}{3} \div 7 = \frac{1}{7} \times \frac{14}{3} = \frac{2}{3}$. Hence, quotient $= 556\frac{2}{3}$

That is, the division is performed as in whole numbers until a remainder less than the divisor is found and then this is treated as any fraction divided by a whole number.

Dividing a Mixed Number by a Mixed Number

There are two ways of proceeding. When the whole numbers are small, the mixed numbers may be changed to improper fractions. When the whole numbers are large and the terms of the fractions small, time is saved by multiplying both dividend and divisor by the least common multiple of the denominators and thus reducing them to whole numbers. Thus,

$$2\tfrac{1}{2} \div 3\tfrac{3}{5} = \tfrac{5}{2} \times \tfrac{5}{18} = \tfrac{25}{36};$$

$$\begin{array}{rl} \text{And} & 345\tfrac{1}{2} \div 16\tfrac{3}{4} \\ & \quad\ \ 4 \qquad\ \ 4 \\ \hline & = 1382 \div 67 = 20\tfrac{42}{67} \end{array}$$

Or, a third method may be used in such a problem as $468\tfrac{15}{17} \div 3\tfrac{1}{2}$. Multiplying both dividend and divisor by 2, we have $937\tfrac{13}{17} \div 7$.

$$\begin{array}{l} 7\overline{)937\tfrac{13}{17}} \\ \ \ 133, \quad 6\tfrac{13}{17} \text{ rem.} \quad 6\tfrac{13}{17} \div 7 = \tfrac{1}{7} \times \tfrac{115}{17} = \tfrac{115}{119} \end{array}$$

COMPLEX FRACTIONS

A complex fraction is merely an indicated division, expressed in the form of a fraction instead of by the division sign (÷), where one or both of the terms (dividend and divisor) are fractions or mixed numbers. Such expressions arise in expressing the work to be done in the solution of a problem or in dealing with a remainder as shown in some of the above divisions. Such fractions, then, involve nothing new and are simplified by perform-

ing the indicated divisions, or by multiplying both terms of the complex fraction by the least common denominator of the fractions composing the terms. Thus,

$$\frac{6\frac{1}{2}}{7\frac{1}{3}} = \frac{13}{2} \times \frac{3}{22} = \frac{39}{44}$$

or

$$\frac{6\frac{1}{2}}{7\frac{1}{3}} = \frac{6\frac{1}{2} \times 6}{7\frac{1}{3} \times 6} = \frac{39}{44}$$

Among the topics often urged for elimination is the topic of "complex fractions," but there seems to be no more reason for this, when a complex fraction is considered as a mere form of expressing division, than to urge that no division must ever be expressed as a fraction. The thing to be urged is that, as in all work, no more complex exercises be given for pure computation than may arise in solving the problems that arise in doing the ordinary life work.

CHAPTER X

DECIMAL FRACTIONS

Decimal Notation

Ours is a decimal-place-value system of notation. Ten units of any order make one unit of the next higher order. Or, expressing the same thing in another way, a unit in any order has one tenth of the value of a unit in the next higher order. The notation, then, of a decimal fraction should be considered as an extension of our notation to the right of *ones' place.* Then the first order to the right of ones' place is *tenths;* the next is tenths of tenths or *hundredths;* the next, tenths of hundredths or *thousandths;* etc.

This notation should be presented by taking some number all of whose digits are alike and asking the pupil the value represented by each digit and by having him compare their values. Thus, in $1111, ask what each of the four 1's represents. Next compare one with another and bring out the fact that each represents one tenth of the value of the one to the left of it. Now, calling any digit a certain unit, get the pupil to see that he then knows what each unit must be, since each has one tenth the value of the one to the left of it.

Next take some number as 3467 and have some pupil read it. Now suppose that the 4 is in ones' place; get the pupil to see that all the other places are known and that 3 *must* then be in tens' place, that the unit of the

place occupied by 6 *must* be one tenth of one or *tenths*, and that 7 must be in *hundredths'* place, for a tenth of a tenth is a hundredth. And thus introduce the use of the decimal point *to locate ones' place.*

Then practice in reading and writing decimals, and mixed decimals should follow until the pupil visualizes the position of each figure as a number is read.

Have pupils see clearly that each zero written between a decimal point and a digit that follows divides the number by 10 for it moves the digit to a lower order. Thus if .5 is changed to .05 or .005, it is changed to a number $\frac{1}{10}$ and $\frac{1}{100}$ as large, respectively.

But if a zero is annexed to a decimal, it does not change its value, for it does not change the order of any of the digits. Thus, .5 may be changed to .500, which is still 5 tenths and no hundredths and no thousandths, without changing its value.

Addition and Subtraction of Decimals

The pupil who understands the fundamental principle of addition and subtraction — that only like units can be added or subtracted — and who understands the notation of a decimal will see that the decimal points must come under each other in order to bring like units in the same columns. Hence, these two processes present nothing new and introduce no difficulties.

Multiplication of Decimals

The only difficulty arising in the multiplication of decimals not already encountered in whole numbers is

the pointing off of the product. This is easily overcome by proper gradation. The work should be rationalized, showing how it follows naturally from the meanings of multiplication by a whole number and by a fraction, already discussed.

Multiplying a Decimal by a Whole Number

Multiplying by a whole number is always an abridged way of finding the sum of a number of equal addends. Hence, 4×7.38 means $7.38+7.38+7.38+7.38$ or 29.52. Thus, it is seen that multiplying a decimal by a whole number must give a product with the same number of decimal places as there were in the multiplicand.

Multiplying a Decimal by a Decimal

Since to multiply by any fraction is to perform both a division and a multiplication, dividing a decimal by 10, 100, 1000, etc., must be understood before developing this phase of multiplication. This division goes back to the meaning of the notation of a decimal. Thus, 5.34 is but one tenth as large as 53.4, for each digit occupies an order one lower in 5.34 than in 53.4. Likewise 5.34 is one hundredth as large as 534. for each digit occupies a place two orders lower. Thus, it is seen that every order to the left to which the decimal point is moved brings each digit down to the next lower order and thus divides the number by 10.

$.4 \times 34.2$ means 4 times one tenth of 34.2. This requires moving the point one place to the left and multiplying the result by 4. But the multiplication may take

place first. Hence, by a few such illustrations, it is easily seen that the numbers are multiplied as if they are whole numbers and then as many decimals pointed off in the product as there are in both factors.

Division of Decimals

The new difficulty arising in division of decimals is the pointing off of the quotient, and this is easily developed by the proper gradation. The first problem should be the division by a whole number, before division by a decimal is considered.

Divisor a Whole Number

Division by an abstract whole number may always be interpreted as the *partition idea* of division, which gives a quotient of the same unit as the dividend. Thus, just as $\$8 \div 2 = \$4$, 8 ft. $\div 2 = 4$ ft., $\frac{8}{9} \div 2 = \frac{4}{9}$; so $.8 \div 2 = .4$, $.08 \div 2 = .04$, etc.

With this understanding of the meaning of division, the pupil who has been taught from the first to write each digit of the quotient directly over the right-hand digit of the partial dividend, from which it was obtained, will experience no new difficulty whatever. Thus, to divide 1833.8 by 53, the pupil sees that, since the first dividend was 183 tens, the quotient was tens, the next quotient ones, the next tenths, and that the decimal point in the quotient must come directly over the decimal point in the dividend in order to show this.

$$
\begin{array}{r}
34.6 \\
53)1833.8 \\
\underline{159} \\
243 \\
\underline{212} \\
31\ 8 \\
\underline{31\ 8}
\end{array}
$$

Divisor a Decimal

The pointing off in this case is more easily developed by changing to a new problem whose divisor is a whole number and thus make the pointing depend upon the *partition idea* as in the preceding case. This will require, first, that the pupils see that a thing could not be divided into, say, 3.2 equal parts; and, next, that multiplying both dividend and divisor by the same number does not change the quotient. The first may be shown objectively and the second by such illustrations as $6 \div 2 = 3$; $60 \div 20 = 3$; $600 \div 200 = 3$, etc., and thus that $17.92 \div 3.2 = 179.2 \div 32$. In the actual work it is not desirable to erase the original decimal points. They may be needed in checking up the original problem. But the new position in the dividend may be marked by a caret to show the position of the point in the quotient.

```
       5.6
3.2)17.9‸2
    16 0
     1 92
     1 92
```

The pupil should see that, when the dividend has fewer places than the divisor, as many or more places may be created by bringing down zeros; for this does not change the value, but merely records the fact that no units of those orders exist.

Changing Common Fractions to Decimals

It is important that the pupil be able to change a common fraction or a ratio to a decimal fraction, but this requires no new fact or process. It comes from considering a fraction or ratio as an expressed division and then performing the division.

There are common fractions with special denominators that are more easily changed by first changing to a common fraction whose denominator is a power of ten. Thus,

$$\tfrac{17}{50} = \tfrac{34}{100} = .34;\ \tfrac{18}{25} = \tfrac{72}{100} = .72;\ \tfrac{112}{125} = \tfrac{896}{1000} = .896$$

The Application of Decimals

As in all subjects, the subject of decimals should be used to meet the pupil's needs if possible. However, his own personal needs rarely ever require a knowledge of decimals, but statistics and data met in other school work and in general reading will no doubt involve decimals if the work is not given too early. The sixth grade seems the most suitable place in the curriculum for the subject. Then such relations as the circumference of a circle to its diameter, the amount of rainfall, the crop's yield per acre, the digestive nutriments of stock feed and of human food, problems involving time and speed, the various constituents of fertilizers, all these furnish a motive for the work in decimals. The use in the subject of percentage, however, is the most common use that the pupil will find for the subject; and, hence, a study of decimals should be followed up by the general discussion of percentage.

CHAPTER XI

PERCENTAGE AND ITS APPLICATIONS

In the past, both textbooks and teachers have presented the subject of percentage as if it were some new topic with its own rules and processes, involving new mathematical principles. The subject was not only presented as a new phase in the development of arithmetic, but it was subdivided into "cases" each with its own rule and formula.

This method of treatment has not entirely passed. We still find textbooks in use that treat the subject in this way; and many of the older teachers, brought up by such a method, still persist in giving it to their pupils as it was given them.

But such teaching is rapidly passing. The pupil is led to see that the subject is merely a continuation of the study of fractions, a new language for an old and well-known idea, and that this change to a new language does not lead to new processes or to new principles.

The Notion and Notation of Per Cent

The term *per cent* must be seen to be a special name for a special fraction, one whose unit is hundredths.

To introduce the *need* of the new term, pupils may be asked such questions as, "If you solve 3 out of 4 of your problems, what part of them do you solve?" "If in an

orchard 7 out of 8 of the trees are apple trees, what part are apple trees?" "If a man in business makes \$15 out of every \$100 he receives for goods, what part of his receipts is profit?" etc.

Lead the pupil to see that it is much easier to say that a merchant makes "15 per cent" than to say that he makes "\$15 out of \$100," or to say that he makes "15 hundredths," and that the term "hundredths" is rarely ever used in business, but that the term *per cent,* which means the same, is used instead. The term is from the Latin *per* (out of) and *centum* (a hundred). So 7 per cent, written 7%, means 7 out of 100, or 7 hundredths.

The Three Problems of Percentage

All problems arising under percentage and its applications fall under three general types: (1) to find a per cent of a number; (2) to find what per cent one number is of another; and (3) to find a number when some per cent of it is known. Examples of these are: (1) Find 25% of 300; (2) 75 is what per cent of 300? and (3) 75 is 25% of what number?

The first two of these problems are by far the most common in business practice, but the third has a place, as will be shown later.

To Find a Per Cent of a Number

When the pupil understands the meaning of the notation for per cent, this class of problems gives no difficulty, for he is familiar with the process involved. He knows that .06×350 means .06 of 350, or that .17 of 285 means

$.17 \times 285$; etc. So to find 18% of 365 means $.18 \times 365$. The only new thing, then, is the notation and the ability to express *any* per cent as a decimal. When the number of per cent is an integer of but one or two figures, as 6% or 35%, the pupil has no trouble in the translation to hundredths; but when the per cent is a fraction, common or decimal, or when it is a whole number larger than 100, the pupil often has difficulty in expressing it as hundredths. Examples of such are: $\frac{3}{8}\%$, .4%, 250%, etc. Errors in changing such per cents to decimals are often made by normal school students and by teachers.

These need not give much trouble if properly presented. .4% means 4 tenths of one hundredth, hence 4 thousandths or .004. The pupil need not go through this rationalizing process for each, but should visualize the expression .4% as "no tenths or hundredths, but 4 in the next lower order"; hence, $.4\% = .004$, $.38\% = .0038$, $1.2\% = .012$, etc., observing that, since % is a symbol for hundredths, when it is removed *two more* decimal places must be used.

When this type is fixed, it is easy to present those per cents expressed as common fractions. Thus, $\frac{1}{2}\% = .5\% = .005$; $\frac{3}{4}\% = .75\% = .0075$; $\frac{3}{8}\% = .375\% = .00375$.

The pupil will easily understand the meaning of those per cents larger than 100 if they are first expressed in the common fraction form. Thus, $200\% = \frac{200}{100} = 2$; $175\% = \frac{175}{100} = 1\frac{75}{100} = 1.75$; $1756\% = \frac{1756}{100} = 17\frac{56}{100} = 17.56$. And thus he sees that, when removing the per cent sign, *two more* decimal places must be pointed off. So $375\% = 3.75$; $560\% = 5.60 = 5.6$.

It is imperative that the pupil can express any per cent

as a decimal before he can find a given per cent of a number; for to find $1\frac{3}{4}\%$ of \$3800 is to find $.0175 \times \$3800$; to find 175% of 396 is to find 1.75×396.

To find a per cent of a number, then, no new rule or formula should be used; but the per cent should be changed to a decimal form and used as a multiplier.

To Find What Per Cent One Number Is of Another

In this problem we are simply asking for the relation of two numbers when expressed as hundredths or per cent. This involves the ratio idea of a fraction; that is, the use of a fraction to express a relation, and the ability to change a fraction to a decimal. A review, then, of these two problems should be given to see that the pupil has the necessary ability to solve this problem of percentage. Thus, if a merchant made \$24 from a sale of \$96, he made \$24 out of \$96 or $\frac{24}{96}$ of the receipts. $\frac{24}{96} = .25 = 25\%$. And, in general, to find what per cent one number, as 325, is of another, as 468, proceed as follows:

The steps are:

1. Express the relation as a fraction.
2. Change the fraction to a decimal.
3. Express the decimal in terms of per cent.

```
     .6944 = 69.44%
468)325.00
    280 8
     44 20
     42 12
      2 080
      1 872
        208
```

This second problem is often treated as an indirect problem, the inverse of the first problem. Thus, to find the result in the problem given above, the reasoning is that 468

multiplied by the answer, if known, would give 325. So the statement is made:

$$x \times 468 = 325,$$
$$x = 325 \div 468 = .6944 = 69.44\%.$$

But this method takes away the relation idea expressed by per cent, is more difficult to understand, and is not related to the processes and principles already known, and is not to be recommended.

It will be observed that this problem requires the ability to express any decimal as a per cent, the inverse of the ability needed in the first problem. So some practice in expressing such relations as the following is needed:

.26 = 26%	1.25 = 125%
.07 = 7%	.004 = .4%
.065 = 6.5%	1.645 = 164.5%

To Find a Number When a Per Cent of It Is Known

This type, usually called the *indirect problem* of percentage, occurs less often than the other two. It is quite commonly urged in current articles on the teaching of arithmetic that this type of problem be omitted altogether, and the recommendation would be well founded if there were no other applications of it than those given in most textbooks. The usual type given is, "If a merchant sold a suit for $24, thereby making 20% of the cost, find the cost." Now, this is not a problem that meets the needs of any one. The answer, if the problem is real, must have been known before the problem could

have been made. This is a sort of "hide-and-go-seek" problem given for "analysis" or "mental discipline" — reasons no longer sufficient when so many real problems are to be found on every hand that meet the needs of man in doing the world's work.

The indirect problem, however, frequently arises in actual business practice. Thus, a manufacturer, knowing the cost of an article and the average overhead charges, etc., based upon the sales, knows that the very minimum price at which the article must be listed must give a certain per cent of profit reckoned on the sales.

Thus, if it costs \$10.26 to make an article, at what must it be sold to give a profit of 24% of the selling price?

Since the cost and profit make up the selling price, if 24% of it is profit, the remaining 76% of it must be the cost. So, if the selling price of it were known and multiplied by .76, the result would be the cost, or \$10.26. Hence, the relation:

$$.76 \times \text{selling price} = \$10.26.$$

Here we have the product of two factors given (\$10.26) and one of the factors (.76) to find the other (selling price). Hence, it is a case of division from the definition of division. The solution then is:

```
        $13.50
  .76)$10.26‸
        7 6
        ----
        2 66
        2 28
        ----
          380
```

Another real application of the same problem would be to find a catalogue price that must be put upon an article in order to discount it at a per cent and yet give a fixed net price. Thus, a merchant may know that \$4.80 is the minimum price at which he can sell an article, but he must put a price upon it from which he can allow his customer a discount of 20% and then get \$4.80.

Since he gets but 80% of the list price, the known relation is $.8 \times$ list price = \$4.80; hence, the list price is \$4.80 ÷ .8 or \$6.

In treating this type, then, the problems should be those of the business world and not those now found in most textbooks, whose only excuse for being there is that they serve as a sort of mental gymnastics.

The Applications of Percentage

The problems given under the topic of percentage are for two general purposes: (1) to develop the ability to see, express, and interpret the relationships between any magnitudes in terms of per cent as encountered in general reading about increases, decreases, discounts, etc.; and (2) to give a social insight into certain phases of commercial life needed for a proper understanding of references met in general reading and in conversation.

Ability to Express and Interpret Relationships in Per Cent

This ability is developed through problems of a general nature relating to magnitudes of all kinds familiar to the pupil and not confined to money transactions as in buy-

ing, selling, loaning, etc. So often a pupil gets a notion that per cent refers to money as "cents on a dollar," or some such false notion.

Hypothetical problems, both direct and indirect, may be used for such development as long as they do not misrepresent facts or give a wrong notion of real practices or conditions.

All three problems may be brought out from the same data; as, "The population of a certain city was 390,300 in 1910 and 468,360 in 1915. (*a*) Find the rate of increase in 5 years. (*b*) At the same rate of increase, what will the population be in 1920? (*c*) If the population increased at the same rate from 1905 to 1910, what must it have been in 1905?"

Business Applications of Percentage

The commercial problems given should be more for the purpose of giving a social insight into current practices in order that one may understand references to them met in general reading and in conversation, rather than to fit for a certain vocation. And yet the methods used should be those used in business and not some stereotyped forms of analysis and procedure so often found in the schoolroom. The problems, too, must be those met in actual business and made from data fairly true to life in order to give true information and not misinformation.

Commission

As a fee for services, salesmen of almost all lines of goods get a per cent of their sales, sometimes in addition

to a fixed salary and sometimes as their only fee for their work. This fee is called their *commission.* Buyers also usually work on the commission or commission plus salary basis. This application, then, is only the first problem of percentage and requires multiplication only. Examples of commission are found on every hand. The real estate agent gets a per cent of his sales as a commission, the agents who canvass from house to house, insurance agents, and salesmen all get a per cent of their sales.

The pupil should get clearly this meaning of commission, which is the meaning he needs in order to interpret references met in general reading or conversation. This may be followed by the work of the commission merchant who sells farm produce, grain, cotton, pork, etc.

Indirect problems found in most of the books of a decade or so ago should be entirely omitted, as they give a false notion of business transactions. The following is a type of such problems: An agent received $1200 with which to buy goods after deducting a 10% commission for buying. Find the commission and the amount spent for goods.

This gives a wrong notion of business practices, for a business firm would not send its agent a fixed amount from which to buy goods after taking out his fee for services.

Discount

In general conversation and reading, the term discount is used to denote a deduction from a former price. Thus, a boy may sell his bicycle at a discount upon what he paid

for it because he has used it a season. A store may give a discount on goods soiled or out of season. A dealer may allow a discount for a cash payment from the price at which the goods are sold on time. The teacher should find such familiar uses of the term as the pupil meets in his everyday life. The chief problem here, as in commission, is the first problem of percentage — to find a per cent of a number. A question, however, may arise as to what per cent a certain fixed discount is. For example, a table marked at $30 may sell for $24 during a special sale. What per cent of discount was given?

The indirect problem can answer no real need, for it answers no question that would arise either in the mind of the buyer or the seller, and hence it should not be given under the subject of discounts.

Commercial discount is a more specialized term and refers to transactions between the retailer and the wholesaler. But the term has a general interest to the average person and should be considered. Thus, it is a custom among certain classes of wholesalers and manufacturers to publish catalogues with a fixed *list price* of their goods and then allow a deduction to the retailer handling their kinds of goods. This is what is known as *commercial discount,* in distinction from the general use of the term.

Usually the list price, or catalogue price, remains the same for long periods; but, when the market changes, new discounts are made. If the market price increases, a smaller discount is allowed; but, if the market price decreases, the discount is increased. The increase in discount is usually made by stating a new discount to be

applied to the previous net price. When two or more discounts to be deducted in this way are allowed, they are called successive discounts.

PROFIT AND LOSS

In the commercial world the subject of profit and loss is one of the important applications of percentage. But for the average person, whether engaged in business or not, the terms are so much used that the subject becomes one of general importance.

This application of percentage may properly bring in all three of the problems of percentage. Thus, a business man may be confronted with any one of the following questions:

(*a*) An article cost me \$48. At what price must I sell it to make a gross profit of 20% of the cost?

Solution. \$48 + 20% of \$48 = \$57.60.

(*b*) An article cost me \$48 and I sold it for \$57.60. What per cent of the cost was my gross profit? What per cent of the selling price?

Solution. \$57.60 − \$48 = \$9.60, gross profit;
9.60 ÷ \$48 = 20%, rate on cost;
9.60 ÷ \$57.60 = $16\frac{2}{3}$%, rate on sales.

(*c*) An article cost me \$48. At what price must I sell it to make $16\frac{2}{3}$% of the selling price?

Solution. Since the gain is $\frac{1}{6}$ of the selling price, the cost is $\frac{5}{6}$ of it. Hence, $\frac{5}{6}$ of the selling price = \$48. The whole of it = $\frac{6}{5} \times$ \$48 = \$57.60.

It must be remembered that, in teaching this or any other practical application of arithmetic, the purpose is to give accurate information so that one may properly interpret business methods and practices. Thus, in teaching profit and loss one should not teach that loss or gain is always reckoned on the cost — a thing taught in nearly all textbooks — for it is not true. There is no uniform agreement among business men as to what should be used as the basis in reckoning the rate of profit. Some reckon the profit on the net cost, some on the delivered cost, and others upon the selling price. No confusion arises, however, if the basis is stated. But the pupil must be led to see that the mere statement that a merchant made 25% has no meaning unless the basis upon which it was reckoned is stated. Thus, one should say, "he made 25% of the prime or net cost," "25% of the delivered cost," or "25% of the sales." Most textbooks make this error. That is, they still give such problems as, "A merchant paid $24 for an article. For how much must he sell it to gain 20%?" This problem is not definite. If one interprets the gain as reckoned on the cost, the solution is: $1.20 \times \$24 = \$28.80$; if it is reckoned upon the selling price, the solution is: $\$24 \div .80 = \$30$. So it should be clear that, if a per cent is to have a definite meaning, it must be followed by a phrase showing the basis upon which it is reckoned.

Simple Interest

The topic of simple interest is so common in conversation and in general reading that it is one of the earliest

applications of percentage to be discussed. It presents no new difficulty and the time element is the only feature not already understood. The topic is taken for information rather than to develop skill in calculating interest or to develop mathematical power. So it is not advisable to develop special methods of computation but rather to teach a general method that follows from the meaning of interest. Hence, the pupil should understand that interest is money paid for the use of money or for an accommodation on an unpaid debt; and that it is reckoned as a per cent of the debt for a year's use of it, even though the interest is collected semiannually, quarterly, or oftener. Thus, the interest of \$600 at 6% is \$36 for a whole year. But for 6 months it would be but half as much, or \$18, and for but one month it would be \$3. In this way it is easily shown that the interest for one year multiplied by the time expressed as a part of a year gives the interest for the given time.

When the interest is for some simple fractional part of a year, as to find the interest of \$1200 at 5% for 4 months, the work may be arranged as follows:

The interest for 1 year being 5% of \$1200 is \$60. But for 4 months the interest would be but $\frac{1}{3}$ of \$60.

$$\begin{array}{r} \$1200 \\ \underline{.05} \\ 3)\underline{\$60.00} \\ \$20.00 \end{array}$$

Where the fractional part of a year is not so simple, all the work to be done should first be written down and all common factors canceled. Thus, to find the interest on \$1500 at 6% for 86 days, the work should be:

$$\frac{86}{\cancel{360}} \times \frac{\cancel{6}}{\cancel{100}} \times \$\cancel{15}\cancel{00} = \$21.50$$
$$\cancel{60}$$
$$4$$

The only problem of arithmetic that concerns the person who is borrowing or loaning money is how to find the interest. Hence, the first general problem of percentage is the only one of the three that is used in interest. The three possible indirect problems still given by some textbooks "for analysis" should have no place in the course, as they never occur in life and detract from the real purpose of teaching the subject, which is to develop a social insight into methods of borrowing and loaning money.

More important than the mere ability to find interest is a knowledge of current rates of interest, the form and meaning of a promissory note, the methods of securing payment, how often the interest is usually collected, etc. While such knowledge is found in most textbooks, those in use may have been written many years ago and some changes may have taken place. So it creates more interest and insures accurate local information if a committee from the class can go to some bank and get the information and bring it fresh to class.

Those problems of the textbook giving rates not in use and with the time element more than one year should not be given, for they give wrong notions as to the price paid for money and also as to the time when interest must be paid. Have pupils see that a man may give a note for

a loan and secure it properly, say by a mortgage on real estate, and not pay the debt for years; but that the interest must be kept paid up as it falls due, which is at least yearly if not more often, as semiannually, this being specified in the note.

Short Methods of Computing Interest

In any business in which much computation of interest is required, as in banks, trust companies, or life insurance offices, use is made of interest tables which greatly lessen the labor of computation. A person outside of such a vocation has so little need of computing interest that it seems unwise to teach more than the general method to pupils in the grammar school.

By the general method is meant: time$\times$rate$\times$principal$=$interest, whether a year's interest is first found and that result multiplied by the number of years, or when all the work to be done is written and common factors canceled.

Since most textbooks give several short methods, two are given here. They follow from first solving the problem by the general method, and thus they help the pupil to see how special cases may lead to special methods.

Thus, to find the interest of \$1750 at 6% for 119 days:

By the general method, $$\frac{119}{\underset{60}{\cancel{360}}}\times\frac{\cancel{6}}{100}\times\$1750=\frac{119\times\$1750}{6000}.$$

Interpreting the result, we see the work to be done may be expressed as follows: To find the interest of any

principal at 6% for any number of days, multiply the principal by the number of days, point off three more decimal places, and divide by six.

Illustration. Find the interest of \$850 at 6% for 70 days. The work is given in the margin.

$$\begin{array}{r} \$850 \\ \underline{70} \\ 6\overline{)\$59.500} \\ \$9.92 \end{array}$$

Had the number of days been 60, then all factors would have canceled except the principal divided by 100. Thus, to find the interest of \$1350 at 6% for 60 days, the general method is:

$$\frac{\cancel{60}}{\underset{\cancel{60}}{\cancel{360}}} \times \frac{\cancel{6}}{100} \times \$1350 = \frac{\$1350}{100} = \$13.50$$

From this the pupil may observe that to find the interest of any principal at 6% for 60 days, point off two more decimal places in the principal.

Compound Interest

When the interest due at the end of any interval is added to the debt and thus draws interest for the next succeeding interval, as in a savings bank, the interest that accrues is called compound interest.

The subject is of chief importance to large investors, as building and loan associations, life insurance companies, banking corporations, etc., who wish to compute the final incomes from reinvesting all interest as it falls due. Such computations are made by the use of compound interest tables.

In teaching the subject the teacher may use it as the basis of a discussion of thrift, the nature and importance of the savings bank, etc. It is always interesting to see how rapidly small regular deposits increase at even a small rate of interest. Thus, $1 per week for 40 years at 5%, compounded annually, will amount to about $6600. And $1 per week for only 10 years at 5%, compounded annually, will amount to nearly $700.

Insurance

The subject of insurance, both property insurance and health and life insurance, is such an important factor in modern life that it should be one of the applications included under percentage, although the term per cent is not very largely used in discussing insurance. The subject is studied, however, for its informational value rather than for the arithmetical problem involved. Canceled or specimen policies should be brought before the class and the subject thus made as real and vital as possible.

Fire Insurance

While there are various kinds of property insurance, as tornado, lightning, burglary, live stock, marine, plate glass, transit, etc., the subject of fire insurance is the most common kind in most communities and may be taken up as a type of all.

Fire insurance is an agreement by a fire insurance company to indemnify the insured against actual losses from accidental fires. The "loss by fire" includes any

damage resulting from chemicals or water used in extinguishing the fire.

The form of policy is regulated by the laws of the various states. It is a fixed form of agreement to do or not to do certain things for a fixed consideration called the premium.

To some policies are attached riders granting certain privileges or making certain restrictions. Among the standard riders the most important is the *average* or *coinsurance clause.* This is an agreement on the part of the insured to carry a certain amount of insurance upon his property or, failing to do this, to become a coinsurer with the company for whatever amount his insurance lacks of the amount agreed upon. Under a coinsurance clause, then, the insurance company pays only such a part of any loss as the amount of their policy bears to the amount of insurance agreed upon. Thus, if a man accepts an 80% coinsurance clause as a part of his contract upon property valued at $10,000, he agrees to carry $8000 of insurance. If he carries but $6000 and a loss occurs, he can collect but three fourths of it from the insurance company.

The rates of insurance depend upon the nature of the risk. Upon a building they depend very largely upon: (1) the location; (2) the construction; (3) the occupancy; (4) the exposure; and (5) whether or not there is coinsurance. The pupils should find the rates upon buildings in various parts of the community and attempt to justify the different rates. Details of local conditions should be learned from some agent in the neighborhood and the problems made to meet these conditions.

Life Insurance

About the only arithmetical problem of life insurance that a pupil in the grades is able to handle is to find the difference in the amounts left the beneficiary by life insurance and the amounts of the premiums at simple or compound interest, knowing the number of premiums that have been paid. But he can understand the nature of the three general forms of policies and also the elements that make up the premium.

The three general forms of policies are: (1) ordinary life; (2) limited life; and (3) endowment.

In the ordinary life policy, the premiums are paid during the life of the insured, and the insurance company agrees to pay a fixed sum to the beneficiary at the death of the insured.

In the limited life policy, the premiums are paid for a fixed number of years, but the guarantee in the policy is not paid until the death of the insured.

In an endowment policy, the insurance company agrees to pay the insured the full amount of the policy if he survives beyond a specified date, or to pay the amount as above if he dies before that date.

The premium that the insured pays is composed of three items: (1) mortality cost; (2) reserve; and (3) expense loading. The first two of these items form the *net* premium and are determined by a given mortality table. *The American Mortality Tables* are compiled from the experience of the New York Mutual Life Insurance Company and are the tables prescribed by statute in

most of the states as the basis upon which the reserve of life insurance companies shall be computed.

The mortality cost is the amount necessary to collect at the beginning of each year to pay the death claims of the current year as determined by the mortality tables.

The reserve element of a premium is the amount set aside from each premium at a given rate of compound interest, usually 3% or $3\frac{1}{2}$%, which will amount to the face of the policy in a given time. So the company's risk at any time is only the difference between the face of the policy and the amount of the reserve.

The expense loading, which is from 15% to 25% of the net premium, is added to meet the expenses of management.

While there are other important things to consider in life insurance, the things discussed here are sufficient to show a pupil the types of policies and the things for which the premiums must be collected. The problems of the subject are too difficult for a pupil in the grammar school. The subject, then, is for information and not for the mathematics involved.

Taxes

The subject of taxes, again, is taken for its informational value, not its mathematical problems. The pupil should understand for what purposes taxes are levied, the way in which the expenses to run a town, county, or state are equitably distributed, and the current rates of taxation. Since the method of levying and collecting taxes varies in different states, the local problems and

rates are to receive the chief emphasis. In such a study of the local methods, the pupil should become acquainted with such terms as assessment, valuation, assessor, collector, board of review, board of equalization, and delinquent taxes.

The tax rates are usually expressed as mills on a dollar, dollars on a hundred dollars, or dollars on a thousand dollars. The general method of finding the rate is to divide the amount to be raised by the assessed valuation of the property.

To facilitate computation, a tax table is usually prepared giving the taxes on sums from $10 to $99, and then the tax on other sums is found by properly pointing off and by addition. The pupils may no doubt be able to get a copy of such a table at the office of the assessor of taxes.

The pupil should see the difference between collecting taxes to meet the expenses of the city, county, and state, and the method of collecting those for the expenses of the National Government. This will lead to a study of duties and customs and the income tax. Here again the mathematics involved is of less importance than the information about the general expenses of the government, where they go, and how they are met.

Banking

Pupils should know the function of a modern bank — that it is an institution where money is deposited for safe-keeping and from which it may be withdrawn when wanted; and also that it is a place where money is loaned

on personal or other security. They should know, too, that most of the money that a bank handles is that of its depositors, and that it is from loaning these deposits that a bank earns money.

In a commercial sense, banks are classified as *banks of deposit, banks of discount,* and *banks of circulation.* All commercial banks exercise the first two of these functions, and national banks the third. The function of a bank of deposit is to receive money on deposit and pay the checks drawn upon these deposits. The function of a bank of discount is to loan money on promissory notes. The function of a bank of circulation is to issue its own promissory notes, which are used as currency. To make this last clear, take before the class some bank note issued by some national bank or federal reserve bank, the only institutions allowed to issue them.

The pupils should also understand the method of opening an account with a bank, the making out of a ticket of deposit, making out and indorsing a check, and the purpose of having a check, that is to be sent out of town, certified by the bank upon which it is drawn, and the transmitting of money by means of a bank draft. All this is vitalized by having pupils gather the information from a local bank.

Bank Discount

The subject of bank discount need not present any new difficulties. It is only a method of collecting interest. When interest is collected in advance, it is called *bank discount.* Thus, if one borrows $300 at a bank for 90

days, the cashier computes the interest at the bank's rate, which is quite generally 6%, and deducts it from the $300 or face of the note. At 6% the interest is $4.50, leaving $295.50, called the proceeds. When the interest is collected in advance, a non-interest-bearing note is given for the loan, for only the $300 is paid when the note is due, not $300 "with interest." Since one paying interest in advance is paying upon the maturity value of the note, it follows that, in discounting an interest-bearing note at a bank, the bank would reckon the interest on the amount of the note when due and not upon the face of the note. Thus, if a note of $1000 bearing 5% interest is to run one year, it is worth $1050 when due. So, if it is sold at a bank 60 days before it is due, the bank will compute the interest on $1050 for 60 days, and not upon $1000. The interest (bank discount), then, is $10.50, leaving the proceeds of $1039.50.

Stock Investments

In the past, the applications of percentage have been hard for the pupil in the grammar school for two reasons: (1) more time has been spent upon the mathematical side than upon the social side, hence the pupil has not had a concrete background for the arithmetical side of the work; and (2) the indirect problem, never arising in business, has been featured for "analysis." If teachers and textbooks would give merely the problems needed to interpret the references met in general reading and conversation, the work would not offer great difficulties.

Interest in the stock market is so general that the

subject becomes one of the important applications, but for its information and not for the mathematics involved.

There was a time when the owner of a business took in one or more partners when he wished to increase the business beyond his own power to supply the necessary capital. But such partnerships often proved unsatisfactory. Each partner was usually held responsible for the acts of any one of the partners. Each change of partnership required new adjustments. As business enlarges, these and many other difficulties arise.

Organizing business under a corporation or company does away with many of the objections of the partnership arrangement. Some of the advantages of a corporation are: (1) one may become but a small investor in several corporations, giving no personal attention to the business, instead of having to make a large investment and giving the business more or less attention as in the case of partnership; (2) one may transfer his ownership in a corporation without affecting the business; (3) the business can more easily expand its capital; and (4) the plan enables the hiring of experts to run the business rather than having it run directly by the partners.

In organizing a corporation a special number of persons interested in starting a business of some sort make application to the Secretary of State of their own or some other state, giving the proposed name of the company, the place and nature of the business they propose to carry on, the amount of capital and the number of shares into which they propose to divide it, and any other informa-

tion required by the law of the state in which they make the application.

The Secretary of State files their application and grants them a permit to sell the capital stock.

When the legal amount has been sold, the buyers of the stock adopt a set of by-laws and elect a board of directors who in turn elect a president, secretary, and treasurer, and other officers from their number. A record of this, showing that the law has been complied with, is then filed with the Secretary of State, who issues a charter, which is an instrument defining the powers, rights, and duties of the corporation.

The capital with which the company organizes divided by the number of shares into which it has seemed desirable to divide it gives the *par value* of a share. The par value, then, is no indication of the real value of the stock, but serves to show what part of the business is owned by the holder of a certificate. Thus, if the $100,000 capital of a corporation is divided into 1000 equal shares, the size of each is $100. So a person owns one thousandth of the business for each share that he owns. This simplifies the distribution of the earnings.

The real or market value of stock is what it can be bought or sold for in open market. Chief among the factors affecting the market price of stock are: (1) the real or prospective earning power of the corporation; and (2) the confidence of the buying public, or lack of it, in the general stability of the enterprise. When the real or anticipated earnings are small, the price is low; when large, the price is high.

The earnings of a corporation distributed among its stockholders are called the dividends. They are declared as a per cent of the capital of the corporation, and each stockholder gets that per cent of the par value of the stock that he owns. Thus, if a corporation, whose capital is $200,000, is to distribute $24,000 in profits, the directors declare a 12% dividend and the holder of each $100 share gets $12. Stock of this kind, whose dividend depends upon the earnings of the corporation, is called common stock. There is another kind of stock sometimes issued called preferred stock, in which the rate of dividend paid is named in the certificate. These dividends, then, become an obligation of the corporation and are deducted from the gross earnings before a dividend can be declared upon the common stock.

There are two types of buyers of stock: the investor who buys and holds for the dividends he will receive, just as he loans money for the interest or buys a house for the rent; and the speculator who buys, expecting the price to advance so that he can sell at a profit.

The problems that naturally arise are very simple when one understands the terms involved. They are: (1) to find how much is lost or gained when buying stock at one price and selling it at another; and (2) to find what per cent of its market value stock paying a certain dividend is earning. These are the only problems with which the general reader or the buyer is concerned.

CHAPTER XII

DENOMINATE NUMBERS AND MEASUREMENTS

I. NATURE AND PRINCIPLES OF DENOMINATE NUMBERS

Formerly the subject of denominate numbers was taught as a distinct topic. Now the work is distributed throughout the grades. The tables are taught as need for them arises in the child's activities in or out of school. A single unit from a table may be taught alone as need for that unit arises without teaching its relation to the other units of the table. Thus, the child may recognize a foot in length and use the measure in finding length without knowing that 12 inches make 1 foot or that 3 feet make 1 yard.

To get a practical knowledge of the tables or the various units, the pupil must see and handle all the common measures. From these units he should make his own table. Thus through measuring he discovers that 2 pt. = 1 qt. and that 4 qt. = 1 gal.; or that 12 in. = 1 ft. and that 3 ft. = 1 yd. It is, of course, unnecessary and impractical to have all of a table found in this way where a very large number is required, as finding how many feet or yards in a mile. After the relation of the various units of a table has been found, through measurement, the tables

should be memorized and used in problems; but just as much of a table as is needed at the time should be learned. Thus, inches, feet, and yards are needed long before rods and miles are needed; ounces and pounds before tons; pints and quarts before gallons or barrels; etc.

All obsolete units of any table and all tables belonging exclusively to a technical education should be entirely omitted.

Reduction of Denominate Numbers

The reduction of denominate numbers involves no new principles. If a pupil understands the meaning of multiplication and division and when to apply these processes to simple one-step problems, he should have no difficulty in reducing units from one denomination to another when need for such a reduction occurs in any practical application.

The old plan of having each table of the weights and measures followed by a lot of exercises for reductions, apart from any industrial or commercial problem, is fast disappearing, and the reductions are now being made when need for them arises in some problem of everyday life.

Such reductions, either *descending* or *ascending*, usually involve one step, or one step and the addition of a lower unit, or one division and a remainder.

Such reductions embody the following operations:

(1) Reduce 5 ft. 8 in. to inches.

5 ft. = 5×12 in. = 60 in.; 60 in. + 8 in. = 68 in.

(2) Reduce 68 in. to feet or to feet and inches.

$1 \text{ in.} = \frac{1}{12} \text{ ft.};\quad 68 \text{ in.} = 68 \times \frac{1}{12} \text{ ft.} = 5\frac{2}{3} \text{ ft.}$

Or, 68 in. ÷ 12 in. = 5, plus remainder of 8 in. not divided. Hence, 68 in. = 5 ft. 8 in.

That is, the last solution is the measuring problem of division. The quotient 5 shows the number of 12 in. in 68 in.; and, since each 12 in. is a foot, 5 is the number of feet.

Avoid such erroneous statements as:

$$\begin{array}{r} 5\text{ ft.} \\ \underline{12\phantom{\text{ in.}}} \\ 60\text{ in.} \\ \underline{8\text{ in.}} \\ 68\text{ in.} \end{array} \qquad\qquad \begin{array}{r} 12\overline{)68\text{ in.}} \\ 5\text{ ft. }8\text{ in.} \\ \text{or} \\ 12\text{ in.}\overline{)68\text{ in.}} \\ 5\text{ ft. }8\text{ in.} \end{array}$$

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION

The four fundamental processes of arithmetic need not be taught as topics and processes of denominate numbers to any great extent. But little, if any, use of such processes occurs in the everyday problems of life. In ordinary measurement, results are expressed by use of fractions in terms of but one unit, instead of two or more before adding, subtracting, multiplying, or dividing them. Thus, instead of adding or subtracting 16 ft. 8 in. and 12 ft. 6 in., one would add or subtract $16\frac{2}{3}$ ft. and $12\frac{1}{2}$ ft. Instead of multiplying 2 bu. 3 pk., one would multiply $2\frac{3}{4}$ bu. or 2.75 bu. Instead of dividing 12 ft. 10 in., one would divide $12\frac{5}{6}$ ft. or 12.83 ft., etc.

II. MENSURATION

If properly presented, the interpretative value of the subject of mensuration is very great. A large number of terms and concepts met in a wide variety of practical activities is acquired. These include the ideas of distance, area, and volume as applied in scale drawing, in computing costs in paving streets and laying sidewalks, in measuring farms and computing productions, in measuring bins for grains, coal, etc., in computing costs of excavations; in fact on every hand are found applications of these fundamental facts of mensuration. And because there is such a rich field of applications, opening up such a possibility of problems of great variety, the topic, like many other topics of arithmetic, is often carried beyond its real need as a tool with which we interpret the physical world about us.

But to have a broad interpretative value to the pupil, he must be led to formulate his own laws and rules and to find an application for them in the life about him. It is not uncommon to find pupils computing the cost of sidewalks of given dimensions as problems found in their textbooks and yet who have no idea of the cost of the walk in front of their own homes or the school, or who are unable even to go out and take the measures and find the cost of such a walk. This sort of teaching is valueless as a means of developing ability to interpret life about them.

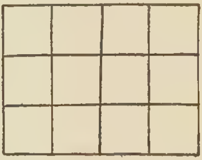
It is only when the subject is taught through having the pupil make his own measurements, constructions,

drawings, etc., and applying the knowledge gained to out-of-school measurements that the subject becomes of real value.

The Area of a Rectangle

The pupil must see that to measure anything is simply to apply some standard unit and see how many times it is contained; and thus to measure the area of a rectangle is to see how many times it contains a chosen square whose side is some linear unit.

Through actual construction the pupil should draw a rectangle whose sides are some integral number of units and then lay it off into square units and count them to find the area in order to see clearly what is meant by the measurement of a rectangle. Through drawing to a scale and the use of squared paper upon which to represent rectangles, he should become familiar with such facts as: the number of squares in a row along any side is the same as the number of linear units in that side;



that there are as many rows, all alike, as there are units in the other dimension; and thus he should see that the total number of square units is the product of the number of linear units in the two dimensions. Thus, the area of a rectangle 4 inches by 3 inches is 4×3 square inches.

Properly developed, pupils will never get the erroneous notion that "inches multiplied by inches give square inches," a thing still taught in many schools. Such a

statement would seem as ridiculous to them as "3 horses × 4 horses = 12 square horses."

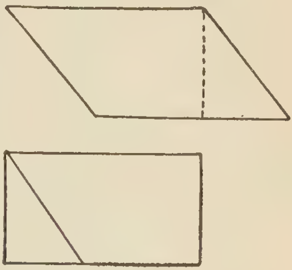
Following the development of the rule or law, pupils should apply their knowledge by measuring actual areas about them. In finding costs, get local prices so that when the cost of plastering a wall, laying a floor, constructing a sidewalk, paving a street, sodding a lawn, or roofing a house is computed the pupil will see about how much of the thing measured can be bought for a certain price; that is, the pupil should know what the sidewalk in front of his home or in front of the school would cost. He should have some idea of the cost to plaster a wall of the schoolroom, or to place the metal ceiling if there is one, or the cost of the blackboards. It is through such work as this that the topic has a real social value.

The Area of a Parallelogram

Through construction, the pupil must see that a general parallelogram whose angles are not right angles cannot be directly laid off into square inches as in the case of the rectangle. Then impress upon him the fact that the rectangle, being the only figure that can be laid off into squares, is taken first and used as a basis, and that all other figures are measured by finding their relation to some rectangle.

The first step, then, in measuring a parallelogram is to reduce it to a rectangle. The teacher's blackboard drawings are not conclusive enough for the child. The part which she removes by erasing and the part which she places in another position by drawing are not the

same parts. From her blackboard presentation of the method of transforming a parallelogram into an equivalent rectangle, the pupil should construct a parallelogram from cardboard, actually cut off a part, and, by rearranging this part, transform the given parallelogram into a rectangle. He need not divide this new rectangle into square inches, for he can deduce the rule from that for finding the area of a rectangle, for he has now discovered that the area of a parallelogram is the same as that of a rectangle having the same dimensions. His attention should be called to the fact that the perpendicular distance between the base and opposite side becomes the altitude of the rectangle and is thus the altitude of the parallelogram; otherwise he may associate the two adjacent sides of the parallelogram with its dimensions.

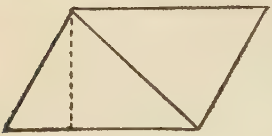


The pupils may not be able to find as many examples of this figure as of the other four plane areas. It is given largely as a basis for the others in order to save converting them into rectangles.

The Area of a Triangle

As in the parallelogram, pupils will see that a triangle cannot be divided directly into squares as in the case of a rectangle. After having thus raised the question of how to measure a triangle, the teacher should have each pupil cut two triangles of the same form and size from

cardboard. But all children should not make triangles of the same size. The teacher should draw various forms and sizes on the blackboard and let pupils use their own pleasure in drawing any shape they wish. Then have each child place his two triangles together so as to form a parallelogram and thus discover that a triangle is just half as large as a parallelogram having the same dimensions. From this fact, he makes his rule. The examples of this kind of figure are more numerous. Have him find areas to be measured that are triangular in form, take the measurements himself, and find the areas.



The Area of a Trapezoid

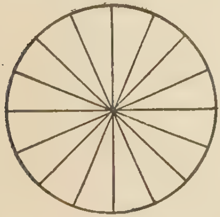
Since the pupil should be shown how to find an area by dividing it up into figures that he has studied, it is hardly necessary to take up the measurement of a trapezoid for any practical purpose; yet it is required by most courses of study and is so easy to present that the teaching of it is not open to serious criticism. As in the triangle, have the pupils each make two trapezoids just alike and place them so as to form a parallelogram. They thus see that, since it took two trapezoids to form the parallelogram, a trapezoid is half as large as a parallelogram having a base equal to the sum of the bases of the trapezoid and having the same altitude as the trapezoid.

As with all other figures, have the pupils look for examples of such figures, measure them, and find the areas.

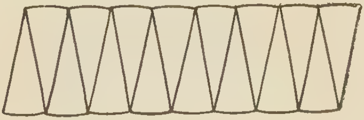
THE CIRCUMFERENCE AND AREA OF A CIRCLE

The pupil does not have to accept the facts and rules for the measurement of a circle upon the authority of textbook or teacher any more than he did the four preceding areas. The rules and relations are as readily demonstrated objectively, and the pupil easily discovers them for himself.

In finding the relation of the circumference to the diameter, use as large circular objects as possible. The pupils may be directed to measure the circumference and diameter of some objects found at home, as the dining room or reading room tables, and bring the data to school. If the measurements have all been carefully taken, the ratios of circumference to diameter in all cases will be nearly enough $3\frac{1}{7}$ to satisfy the class that the ratio is constant. The more exact ratio, $\pi = 3.1416$, should be given and used where great accuracy is required.



A



B

To discover the area, have pupils each cut from cardboard circles of various sizes, not too small, and divide them into at least sixteen equal parts as in figure *A*.

Have them take half of these parts and arrange them as in the lower half of figure B, and then take the other half and fit them into these, completing figure B, and thus discover that the area of a circle is equal to that of a parallelogram whose base is half the circumference and whose altitude is the radius. Then, $A=\frac{1}{2}C\times R$. But $C=\pi D$, hence $\frac{1}{2}C=\pi R$. Then $A=\pi R\times R=\pi R^2$.

Find all the local applications possible.

The Volume of Rectangular Solids

This is the fundamental volume from which the measurement of other volumes is deduced. The same plan of comparing the volume with some standard unit, suggested in the measurement of a rectangle, should be followed here. Or, instead of dividing a solid into cubic units, a box whose inside measure is some integral number of units may be filled with cubic units. The pupil should see that the number of cubic units along any dimension, as the length, is the number of linear units in the length; the number of such rows in the width is the number of linear units in the width; and that the number of such layers in the whole volume is the number of linear units in the height. Thus, if there are 6 inches in the length, there will be 6 cu. in. in a row along the length. If 4 inches wide, then there will be 4×6 cu. in. in a layer; and, if 5 inches deep, there will be in the whole volume $5\times4\times6$ cu. in. or 120 cu. in. If the rule is developed in this way, the pupil will not use such erroneous statements as 5 in. $\times$ 4 in. $\times$ 6 in. $=$ 120 cu. in.

Follow up the development by local problems, as ex-

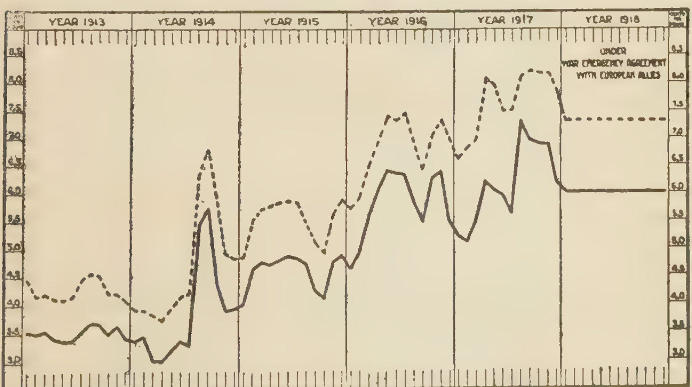
cavations for cellars, capacity in tons of coal bins, in bushels of grain bins, the air space of the room, the snow removed from walks, or in any grading or constructing work that is being done in the neighborhood.

Other Volumes

The illustrations already given serve to emphasize the method of teaching measurement. Any good textbook gives diagrams to illustrate the development. But with children, diagrams of the textbook are not sufficient. The objects illustrated by the diagrams should be used.

GRAPHIC REPRESENTATION OF STATISTICS

The relations between magnitudes is more easily grasped by most people if expressed by graphs rather than by figures to represent the number of units. For this reason newspapers, magazines, trade journals, and all kinds of circulars and pamphlets make very frequent use of the

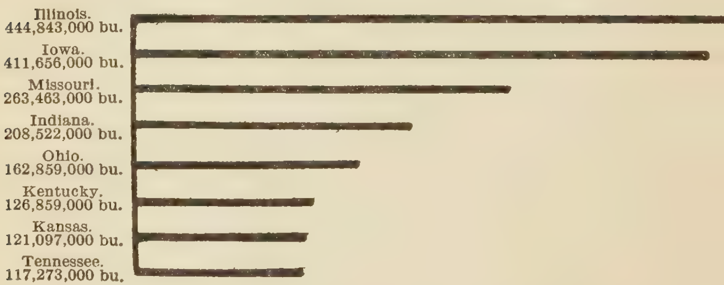


Market Quotations—Raw Sugar and Refined Granulated Sugar—Years 1913, 1914, 1915, 1916, 1917, 1918—Comparing and Showing Refining Difference
Raw Sugar ——— (Duty Paid Price) Refined Granulated - - - - (Net Wholesale Price)

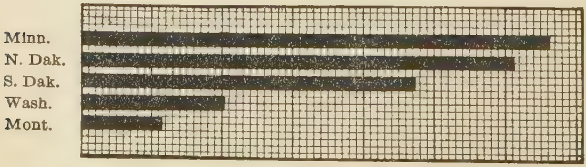
graph where magnitudes are shown in comparison with other magnitudes or with their former values. Thus, the graph on the previous page is reproduced from an advertisement of the American Sugar Refining Company.

There are three general forms in use depending upon the nature of the comparisons to be made.

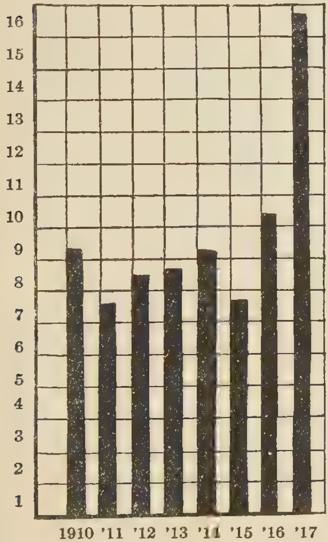
In comparing the productions, wealth, population, etc., of different states or countries, straight lines are generally used. Thus, the comparative production of corn in the following states, 1917, could be represented graphically as follows:

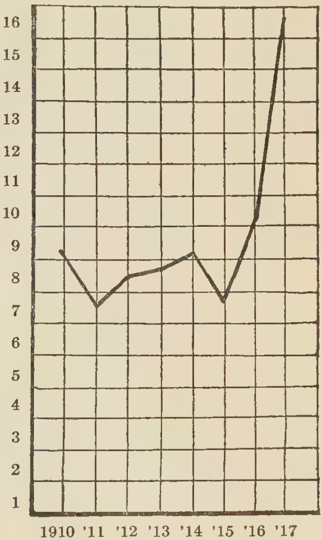


For the construction of such graphs, squared paper is very convenient. Thus, the following comparison of the wheat crop of 1917 in the five leading wheat states may be shown as follows:



To show the relative changes of the same magnitude through a given period of time, both the straight line and the broken line graphs are used. Thus, the variation of the September prices of hogs ranging over a period of eight years may be represented in either of the following ways:

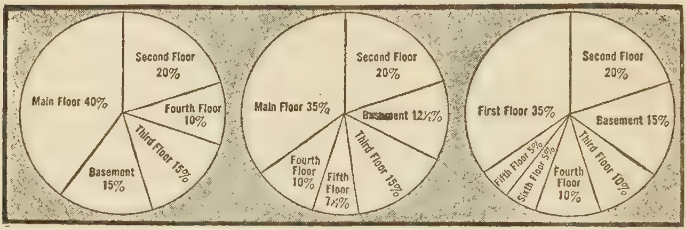




To show the relation of a part to the whole, a circle is more often used. Thus, the following graph shows how the rent was apportioned on each floor of three stores during a recent year. It is a portion of a chart given in *The System*, December, 1917. (See page 162.)

Pupils should be encouraged to bring in graphs found in general reading and interpret them. It will be found

that there is no rigid adherence to a certain form for a certain type of comparison, but that the uses shown here are general.



Enough work in the construction of the three kinds shown here to readily interpret those found in general reading should be done if time permits.

CHAPTER XIII

THE PURPOSES AND NATURE OF PROBLEMS

The great advances made in the teaching of arithmetic during recent years have come very largely through an improvement in the nature of the problems. This improvement has followed from the broader conception of the purposes of a problem discussed in this chapter, for a problem is no longer considered a mere tool to train the mind.

In the lower grades the problems differ greatly in their nature and purpose from the problems of the upper grades. For that reason problems will be discussed under two general divisions: the problems of the primary grades, and the problems of the upper grammar grades.

The Problems of the Primary Grades

While the child in the first four or five grades may get several things as by-products of his study of a problem, such as the habit of looking upon the quantitative side of life, an appreciation of the value of arithmetic in doing the world's work, something of a social insight into certain phases of life, etc., the chief purposes of the problem during these years are: (1) to clarify or rationalize the

facts and processes; and (2) to motivate the pure drill work in the fundamental processes. This leads, then, to an inquiry into the nature of problems that will do this.

Clarifying a Process

We are beginning to realize that the child's own thought and activity should dominate classroom instruction; that all knowledge, to be real, must be based upon the real experiences of the individual learner; and that arithmetical ideas must have a background of mental images which are the outgrowth of real experiences, in order to exist. Hence, all facts and processes should be presented objectively or through very familiar situations. Through such teaching the facts have a real background of vivid mental images and the pupil is able to use them in the situations that arise in his real life. The chief source of failure of a pupil to apply his arithmetical facts is that they have been put before him as mere verbal facts without any concrete basis.

When a problem, however, is given to clarify a fact or process, it must be made from real objects seen and handled by the class, from the pictures of such objects, or from experiences that are very familiar to the pupils.

It is through developing the facts of addition, multiplication, etc., through such concrete situations, that the pupil gets the real meaning of them and thus knows how to use them in new situations that arise. If the first problems are made up about real things that the child sees, then next about pictures that call up real objects, and finally about things that he easily images, he grows

into the real method of approaching the more difficult problems that he will meet later in the course.

Motivating Drill Work

Aside from rationalizing a fact or process, the problems given during the early years of a child's school life are given very largely to furnish a motive for, or an interest in, adding, subtracting, multiplying, and dividing. Such problems should require little, if any, thought in order to discover the process to be used.

To motivate the drills, however, the problems will have to be more attractive than mere drill work from abstract numbers. This will require that the answer is really wanted because it meets some personal need of the pupil, adds in some way to his pleasure, or satisfies his curiosity.

There are at least five general types of problems that may be made to meet these conditions:

(1) Those meeting some personal question; as, "If you are saving money to buy a coaster costing $5.75 and now have $3.90, how much more will you need?"

(2) Those having to do with some home affair; as, "If your father uses 12 tons of coal each winter, costing $7.25 per ton, how much does he have to spend for coal?"

(3) Those about some neighborhood activity, as costs of keeping up the parks, playgrounds, schools, etc.

(4) Those that appeal to his civic pride, as those comparing the growth of his town or state with some other, or those about some industry in which his town or state excels.

(5) Problems in which the answer simply appeals to his curiosity, as the number of things, such as movie tickets, baseball bats, etc., that could be bought for the money that it costs to run the government for a day, or with the wealth of some of the well-known men of wealth. These furnish larger numbers than those of the real problems discussed under the first four heads.

The Wording of a Problem

In the primary grades, as has just been pointed out, the child's present interest is the fundamental factor to be considered in selecting or making problems. The ultimate use of the subject in adult activities has little, if any, place in such a selection in the lower grades. That a child may grow into the true method of solving a problem by comparing the magnitudes involved and thus discovering whether they are to be added, subtracted, multiplied, or divided, the early problems should be "story problems" about objects present, then about objects owned by the children or familiar to them.

Not only should problems be about familiar objects, but they should be problems that the child might very probably ask. Thus, "Who weighs more, you or I?" would be a problem that a child would be much more likely to ask than, "How much do you and I both weigh?"

In the early years, the interest is determined much more by the wording of the problem than by the numbers and processes involved. Thus, a "story problem" such as, "Mary picked 8 pink roses this morning. She is going to leave 5 at home for her mother and take the rest

to her teacher. How many can she take to her teacher?" is much more interesting to a primary pupil than the same problem stated as, "A farmer had 8 sheep and 5 of them died. How many were left?" or worse yet, "Eight horses less 5 horses are how many horses?"

It would be much more interesting and do much more toward developing a proper method of solving problems to give the problem, "If you had put 9 little rabbits in a pen and, when you went to feed them, found that there were but 6, how many were gone?" than to give, "A man had 9 cows and sold 6 of them; how many were left?"

It is much better to say, "Frank got 50 *Saturday Evening Posts* this week and sold all but 4 of them; how many did he sell?" than to say, "A farmer had 50 acres of wheat and he harvested all but 4 acres; how many acres has he harvested?"

In making problems, then, the teacher must assure herself that the objects are very real to the children and the problem is so woven into a story that they get the correct mental picture that she describes.

The Problems of the Upper Grammar Grades

There are some who seem to consider that the development of ability to compute is the final end of a course in arithmetic. But skill in computation is but the means to an end. Those who would try to meet the demands of the business world for a better product of the schools in the subject of arithmetic by merely emphasizing drill in computation are following a course about as inadequate for the purpose of developing a number sense or a mathe-

matical type of thought as a training that confines itself to forming the letters of the alphabet would be as a final preparation for the career of journalism.

The final aims in a course in arithmetic are:

(1) To develop *power* in the student to see and to express the quantitative relations that exist among the magnitudes that come within his experience, and to interpret the numerical expressions of such relationships;

(2) To develop in the student the *habit* of seeing such relationships, especially those vital to his present or future welfare; and

(3) To give the student a *social insight* into current business and industrial practices through which he can interpret references, met in general reading and in conversation, to the world's activities.

These three aims, of course, come through the problem side of arithmetic.

Power to See Relations

The task of developing power to see and to use quantitative relations is a most difficult one. When we think we are developing such power we are often merely storing the memory with rules, forms, and processes. If the wording is changed, pupils are often at sea as to how to proceed. To develop power, the problems and the wording must be so varied that the solution has to depend upon a rational analysis of conditions and not be a mere act of memory. This power can never come through blind adherence to rules and stereotyped forms of solution made to fit artificial types of problems.

The first requirement of a problem to meet this purpose is that it be concrete to the pupil. Before he can analyze it and thus discover the steps of the solution, he must have a very clear comprehension of the facts — an accurate mental image of conditions.

The problems must not only be very concrete, but they must suitably task the pupil's powers. However real a problem is, a pupil is much more interested in it if it is difficult enough to require some thought in order to determine what to do. A pupil soon tires of long lists of problems all solved alike where he gets no exercise except in computation. Teachers will find that without thought-provoking problems interest will soon lag.

While problems must always be concrete, they need not always be real. In the development of this power to see quantitative relations, the hypothetical problem still has its place. These problems, however, must never be such as to give wrong ideas of business practices. One does not have to resort to the old "hare and hound" type of problems of the past nor develop wrong notions of business practice to get problems that furnish valuable training of power to see and express relations. The data may be worth while and the problems those that may arise in life. For example, if one is trying to develop the power to see and to interpret the relationships expressed by per cent, he may begin with a certain bit of data and bring in all the various relations without giving any wrong impression of business practices whatever. For example, let it be given that it cost $38.45 to manufacture a certain gasoline engine, and the cost to sell it amounted to $15.38

more. Now, the following questions may be asked of an advanced class:

1. The cost to sell was what per cent of the cost to manufacture?

2. The cost to sell was what per cent of the total cost to manufacture and sell?

3. At what must it be sold to give a profit of 2[illegible]% above the cost to manufacture and sell?

4. At what must it be sold to give a profit of 25% of the selling price?

5. At what price would it have to be listed in order to allow a discount of 40% from the list price and still give a profit of 20% of the total cost to manufacture and sell?

6. At what price must it be listed in order to discount the list price $33\frac{1}{3}$% and 10% and still make a profit of 20% of the actual selling price?

And thus one may go on making up a lot of questions of increasing difficulty to meet the maturity of the pupil's powers, that have great value in bringing out the relations expressed by per cent. While one hears a great amount of criticism of any but "real" problems, it ought to be clear to any teacher that such criticism is not based upon a careful consideration of the aims to be attained.

The Habit of Seeing Relations

One of the greatest sources of waste in so much of our teaching is that pupils are not so taught that they can apply the knowledge gained in the classroom to the work found outside of the classroom. There should be a constant utilization of outside experiences to clarify and

rationalize the work of the schoolroom, and likewise the facts learned in the schoolroom should constantly be used to interpret conditions met without the schoolroom. Every subject in the textbook should be supplemented by "community problems" made from data gathered by the teacher and pupils from the immediate community in which they live. It is only through making such problems that pupils develop the habit of looking upon the quantitative side of life. Too often the problems of the text mean no more to them than mere assignments in order to see if they can get the answers. However, with the improved type of problems that are beginning to find their way into our newer texts, a teacher can do a great deal toward developing proper habits by making the right use of them. But the pupil must be taught to picture the situation and to ask himself if the conditions of the problem meet local conditions, whether the answer is a reasonable one, etc.

Problems of thrift are among the most useful in the development of the habit of seeing those quantitative relations vital to our welfare. Thus, such questions may be: How much is saved by buying enough potatoes for the winter direct from the farm at digging time instead of buying them by the peck from the grocer during the winter? How much is saved by buying canned goods by the case instead of by the can? How much is saved by buying material and making your own dress instead of buying it ready made? Will it pay better to rent a house or to own one? Will it pay in your town to build a house for rent? About how much could you expect to

make net upon your investment? Has the price upon vacant lots in your town gone up so that the one holding them for a certain time made or lost? In fact, there is no end to good problems that may be made from the material found in any community.

Problems that Contribute to Social Insight

As the pupil's social world expands, he is carried from problems dealing with his own personal affairs and local situations to those of more general interest. However, the teacher must have a much broader concept of such problems than that they are to meet the demands of the commercial and industrial world. They should be problems that contribute to the pupil's interpretation of the social world in which he lives — that contribute to "a broadly socialized and modern culture," not a mere vocational training. That is, a study of arithmetic should enable one to interpret references met in general reading and in social intercourse to all the common affairs of life, as taxes, insurance, interest, stocks, bonds, notes, mortgages, and other business forms, and to any references to measurements and relations, to the statistical relations represented by graphs, and to any other phases of the subject through which he can better interpret the quantitative side of his environment.

To do this, the problems must be *real*, true to present-day conditions, and not fictitious problems about real things. But however "real" they are, they must be full of human interest to the child. The following problem given in a recent textbook for third grade pupils may be

real, but it lacks any element that makes it interesting to a child of nine years. The problem is: "If a 200 pound sack of fertilizer contains 8 pounds of nitrogen, 10 pounds of potash, and 16 pounds of phosphoric acid, how many pounds of these plant foods are there in a sack?"

Simply because a problem is "real" to one in some specialized vocation, it does not follow that it necessarily contributes to the social insight of the pupil. Unless the problem is one likely to come up in interpreting some phase of life which is of interest to the student either at the time or at a probable future time, it is useless from this standpoint.

It is not to be understood that a problem meets but one of the three aims of a problem at a time. A good problem may, and often does, meet all three at once. The problems to meet the last condition discussed will in general meet this one. The necessary conditions are that the problems answer some question arising from some social issue, and that the data be real and of enough interest to warrant their use. Such problems are as varied as the activities of life. Those of greatest value are the ones having to do with industrial and commercial terms and practices, but they are not all included in these topics. They may relate to all the various productions needed for the world's comfort — the amount, the value, the means and cost of the distribution of them, and so forth.

CHAPTER XIV

THE ANALYSIS AND SOLUTION OF PROBLEMS

Difficulties in Teaching

It is the experience of teachers that to teach a pupil how to interpret and solve a problem is a much more difficult task than to develop rules and train for skill in computation. And it is not strange that this is so. The formal side — the methods of computation — is merely a machine which the pupil learns to operate, and it works the same upon all occasions and under all conditions. But the solution of a problem is an application of judgment. The problem must be analyzed, the conditions must be carefully studied, the magnitudes wanted must be compared with those given, and an act of judgment tells what processes are to be used. The solution of a problem cannot be done by rote, as the process of addition or multiplication can be done. The solution can be discovered only through a contribution from the experiences of the individual and an act of judgment. And, hence, if the necessary experiences are lacking, the problem cannot be solved.

It is obvious, then, that no rule can be laid down that will teach a pupil how to solve a problem; but a discussion of the sources of failure may prove profitable to the teacher.

Why Problems Are Difficult

When a problem is difficult to any large number of a class, the teacher cannot lay the blame to the dullness of the children. A large per cent of our pupils are capable, normal children. Then we must look for the trouble either in the material we are using or in our method of presenting it. If we examine the material found in our textbooks, we shall find much that may be very justly condemned. While there has been a very marked improvement in our textbooks in recent years, many of the problems still have but little meaning and but little appeal to children. Many of the problems designed for the primary grades are taken from the adult's world and not the child's world. They are about business, manufacturing, transportation, and various industries with which children have no vital contact, instead of problems about the child's play, his daily activities about the home, or his constructive activities. In so far as textbooks have poorly chosen their problem material, they have contributed directly to the trouble in question and hence are at fault. But the material in the textbook is not the only available material. A resourceful, wide-awake teacher who understands the child's interests and needs and who understands the fundamental principles of teaching will be able to obtain admirable results in spite of a poor book.

A Fundamental Principle of Teaching

At the very root of all successful teaching lies the fundamental principle that all knowledge possessed by

the individual to be real and permanent must be grounded upon and developed out of his real experiences.

And thus all arithmetical ideas must have a background of mental images in order to exist. These mental images in turn must be the outgrowths of some form of experience. Hence, the real ideas which constitute the child's arithmetical knowledge must be developed through the individual experiences of the learner which have been provided for either in or out of school. The failure to provide these clear mental images, which the statement of a problem should call up in the child's mind, is perhaps the chief source of all failure to solve a problem.

Concrete Problems

Another way of stating this source of failure to solve a problem is to say that the problem was not concrete to the pupil; that is, he was not able to form a clear mental picture of the situation described. Unless such a picture can be formed, the pupil is unable to make a comparison of what is wanted with what is given in order to form a judgment of what to do to obtain the desired result.

When a pupil fails to form such an image, it may be that he has not read the problem carefully. Or he may have read it carefully and yet, through lack of experience in the situation described, he may not be able to form a picture of the situation. Recently, a little girl in the fourth grade told me that her pencil cost five cents. I asked her how much two such pencils would cost. She told me at once, of course, and then she reminded me that she was in the fourth grade and that such problems were

too easy for her. So I said, "Well, try this one: If a meter of silk costs five francs, how much will two meters cost?" She could not tell me. She said, "I don't know what a meter is and I don't know what francs are, so of course I can't solve it."

Some months ago I was trying to explain to a young teacher that the quotient of the length of a row, divided by the distance between plants, did not give the number of plants in the row; but that the quotient was merely the number of divisions into which the plants divided the row, and hence there must be one more plant than division. After I had exhausted my stock of word pictures, I took a pencil and began a diagram, when the teacher to whom I was explaining the problem exclaimed: "Oh, I see it. It is just like hanging handkerchiefs on a clothesline. It takes one more clothespin than there are handkerchiefs." I had finally called up some experience in her own life through which the problem became concrete to her.

Sometimes the largeness of the magnitudes involved takes away the concreteness of the situation, and the mind sees but the empty figures without any concrete basis back of them. I recently saw a student attempt to find the capacity of a bin, having been given the dimensions and also the statement that one cubic foot was equal to .8 of a bushel. After correctly finding the number of cubic feet in the bin, he divided by .8 to find the number of bushels. But the teacher said, "What part of a bushel does one cubic foot equal? Then what does 2 cubic feet equal? What does 3 cubic feet equal?"

These were correctly answered; then the pupil saw clearly that he should multiply .8 of a bushel by the number of cubic feet in the volume in order to get the capacity in bushels.

The only true aid that the teacher can give is to help the pupil translate the problem into his own experiences. Power to do this cannot be given by any book upon "method," but depends upon the personality, versatility, and experiences of the teacher. Teachers need to get as wide a contact as possible with that side of life to which arithmetic is applied in order to give practical and concrete applications to every topic of the subject. It is only through such knowledge that the most efficient work can be done in the application side of the subject.

Book Problems *vs.* Local Problems

The problems of any textbook must of necessity be designed for pupils of all communities. They must draw upon the experiences of all children alike. They cannot take into account the different environments of children of different localities. Evidently, then, all problems cannot be equally concrete to all children. Hence, it follows that the textbook should be supplemented by problems taken from the child's own local environment. These problems are not only more concrete to him but, if dealing with affairs in which he is interested, they furnish a much stronger motive for learning arithmetic than do those of the book. These local problems may deal with personal or family purchases, with the child's constructive activities, with local stores and markets, econ-

omy in buying in quantities, local improvements, farm crops, etc. The data for these problems should be gathered by both teacher and pupils, and much of the problem making should be done by the pupils themselves. In this way arithmetic provides a much better equipment for life than can be obtained from the textbook alone, for the problems encountered in real life are not in general formulated for us; but from known data the individual must formulate his own problem. In so far as possible, the school should provide conditions for incidental problem work in the various school activities. These problems, however, must be "unavoidable" problems that arise and *must* be solved. In other words, they should not be problems that are assigned merely in order to have a problem that will give drill in certain processes, but the problems must be those in which it is *necessary* to know the answer and not problems giving results for which there is no need or interest. The making of "number stories" by the children is a beginning in this important phase of the problem work.

Other Sources of Failure

While no doubt the chief source of failure to solve a problem is the lack of a clear-cut mental picture of the situation described, there are other causes that should be discussed. If a teacher will list the reasons why a pupil fails to get the correct answers to assigned problems, she will probably find that the failures will fall under the following general heads:

1. The problem was not concrete.

2. The computation was inaccurate.

3. Approximations were made before the computation was completed.

4. The meaning of the fundamental processes was not understood.

5. The author of the problem assumed facts that were not known by the pupils.

6. The problem was too difficult; that is, the relations existing among the data were too complex. In other words, the pupil lacked the mental power to reason out what processes to use.

Inaccurate Computation

The lack of concreteness has already been discussed. Perhaps errors in computation are as frequent a cause of wrong results as lack of concreteness. And yet this source of error may be practically eliminated. From the very first step in the teaching of the written processes, the pupil should be taught to check his work. He should feel from the first that when a computation has been performed but once it is but half done. It must be reviewed or checked in some way to insure its accuracy. This is done in every vocation. However small the amount, a clerk always checks every sale-slip. To make one error out of a thousand, a clerk would lose his position. Yet a pupil is allowed to hand in work without checking it and, if he gets but two out of ten answers wrong, he is given "80%" or "good." Some schools are taking as their motto: "One hundred per cent accuracy in all computation is our aim." This does not mean

that pupils are never expected to make a mistake but that, just as in business life, their computations are to be checked until they know when work is turned in that it is 100% accurate.

But this habit cannot be developed as long as teachers call the work "good" when two out of every ten answers are wrong. Higher standards will have to be required in order to furnish a motive for checking the computation.

Error in Approximating Results

Even if all computation has been carefully checked, a pupil may fail through making an approximation before the computation is completed. Pupils should be taught to see clearly just what effect an approximation during the computation is going to have on the final result. Recently a pupil was "just sure the answer book is wrong" to the following problem: "How many cubic feet of water can a V-shaped gutter discharge (flowing full) in a day if it is 8 inches deep and 16 inches wide at the top, and the water flows a foot a second?"

He had reasoned correctly and his computation was correct as far as he had carried it. He saw that the problem was to find the volume of a triangular prism whose base was the cross-section of the gutter and whose height was $60 \times 60 \times 24$ feet. His error came from first reducing the cross-sectional area (64 square inches) to a decimal part of a square foot "true to thousandths" before multiplying by 86,400. He did not realize, however, that the error in approximating the third decimal was multiplied 86,400 times in his answer.

Pupils should be taught that, when a solution consists of multiplications and divisions only, all work should first be indicated, then all common factors canceled from both dividend and divisor, and all multiplications in the dividend performed before any of the division is done. Had this been done, the problem given above could have been solved without any use of a pencil except in the cancellation.

Forecasting Results

After a pupil has "thought out" what to do in the solution of a problem, he should be required to *estimate the result.* Such an estimate serves as a check upon absurd errors in computation, such as misplacing the decimal point, failing to multiply by all factors, etc.

But the habit of making close approximations has a very much more important use than merely to help check up the work of computation. In practical life, a mere approximation of a result often suffices. So, if one is trained to make rather close approximations without a pencil, he is getting a training in a very practical phase of the subject — power to see the world about him from the quantitative standpoint.

Meaning of the Processes Not Known

It sometimes occurs that a pupil does not know what process to apply because he does not know what the processes mean. I once asked a little girl in the third grade the question, "Five threes make how many?" She looked at me in bewilderment and then asked, "Do you

mean *and* or *times* or *less?*" She would no doubt have had trouble in applying her facts to the solution of problems.

Not understanding the meaning of a process, or not seeing just what it means when applied to a given situation, often leads to error. In the problem, "How many plants 8 inches apart can be set in a row which is to be 20 feet long from the first to last plant?" the pupil usually divides 240 inches by 8 inches and answers "30 plants," not seeing what the division actually means — that it is finding how many times the row will contain an 8-inch measure and that, from the number of times it is contained, the number of plants must be found by visualizing the act of placing a plant at each end of the measure the first time it is applied and then one plant for each new space measured off.

In the problem, "From a sheet of Manila board 22 inches by 28 inches, how many cards 8 inches by 11 inches can be cut?" the pupil often answers "7." He jumps at the conclusion that it is one area divided by the other and does not see that this is not the same as actually laying off the sheet into cards, in which case he would find that he could get but six.

Necessary Facts Not Known

Sometimes a pupil has the correct picture of the situation described in a problem, but lacks a knowledge of the facts upon which the solution depends. This failure occurs most frequently in the problems of mensuration where the pupil does not know how to find certain areas or volumes. Thus, I recently saw a pupil attempt the

solution of the following problem: "The cross-section of a railroad tunnel 132 yards long is in the form of a 16-ft. square surmounted by a semicircle. Find the number of cubic yards of earth removed in its excavation." The pupil stepped to the blackboard and sketched accurately the picture. She stated that the problem was to find the volume of a prism and half that of a cylinder and add them. She then failed in the solution because she tried to find the volume by multiplying the perimeter of the cross-section by the length of the tunnel. In a case of this kind, there is nothing to do but re-teach these fundamental rules. It is not sufficient to ask the pupil to "review the rules for to-morrow," but the rules should be again presented as concretely as possible. Pure memorization of facts without the mental image of personal experiences as a basis is almost valueless. If all rules of mensuration and all other facts of arithmetic are presented objectively, they are much more easily retained than when learned by rote. The vivid mental image growing out of an experience is much more lasting than a fact memorized through the repetition of meaningless language.

Complexity of Relations

Sometimes a pupil cannot solve a given problem because the relations existing among the data are too complex. That is, failure may be due to lack of sufficient power to reason. However, pupils often seem to fail from lack of reasoning power when the real trouble is that they have made no serious attempt to analyze the

situation and thus to discover the essential relations that exist. They depend too largely upon some word or words in the description of the situation, and then, through memory alone, associate the problem with some one encountered in the past experience for their "cue" as to what to do, rather than make any serious analysis of the conditions described.

When the answer to a problem is known, rather than reason out the solution, the pupil is very apt to try to juggle the figures in some way that will give the result. Recently I saw a pupil attempt the following problem: "If the interest of \$500 for $2\frac{1}{2}$ years is \$50, what is the rate?" She knew that the answer was 4%. Her solution was as follows:

$$2\tfrac{1}{2} \times \$50 = \$125; \quad \$500 \div \$125 = 4\%$$

In most schoolrooms, the teaching of a systematic method of attack in the solution of a problem is greatly neglected. The assignment of "the next ten problems" with no thought given as to their fitness for the class is entirely too common a practice. The recitation should not be a place in which the teacher is to find out whether assignments have been finished, but it should be a workshop in which the pupil is taught to translate the arithmetical question into terms of his own experiences.

Home work should be given but sparingly and should consist chiefly in performing the mechanical computations involved after the "what to do" has been discussed in the classroom. When pupils are assigned home work that they do not understand and have to depend upon

help given by the parent, there is no uniformity of the instruction of the various members of the class; and, in general, the home instruction differs from that given by the teacher. Hence, confusion arises. But worse than that is the fact that much of the home work brought in by the pupil is not his own but that of his parents, and hence the teacher, knowing that the work is brought in, does not realize the difficulties that the child has encountered or that he does not fully understand the work.

The classroom period should be given up to instruction — to real teaching — and to rapid motivated drills. The working out of mechanical details, as computing results, etc., should be the only part left for home work, in most grades at least.

It is a common experience of teachers that pupils who can solve problems when stated in the familiar terms of the textbook in certain stereotyped forms of expression fail to solve the same problems when stated in language less common or when met in actual life. To overcome this defect demands a more varied use of language in the description of a situation as well as a wider range of material used.

To develop a systematic method of attack in solving a problem, the pupil must be led through simple types of one-step, two-step, and three-step (or more) problems, all stated in various ways and descriptive of a wide range of material, all within the range of his own experiences and interests. He must state the problem in his own words, pointing out just what is wanted and what is given in the problem that will help find this. Then, from a

clear mental image of the things involved, he will see the essential relations that exist and will know what to do. The numbers involved in any new problem should be so small and the computation so easy as not to retard thinking.

The Solution of Problems

The solution of a problem requires an act of judgment and cannot be made merely the application of a rule or formula. Hence, there is as large a variety of solutions as there are variations in the nature of problems.

After reasoning out what to do, the computation should be made as simple as possible. When there are several steps in the solution, it is best to reason entirely through to the end before performing the computation. Sometimes one step will balance or cancel a later one and thus work is saved. For example, to find the cost of 18,760 lb. of hay at \$18 per ton, it is seen that to divide 18,760 by 2000 gives the number of tons, which is 9.38. Then 9.38×\$18 gives the cost of all. But observing the processes to be performed, we see that by multiplying 18,760 by 9 and pointing off three decimal places the work can all be done without a pencil. If the work is to be written down, it should be done in such a way that the pupil does not think that he is multiplying pounds by something else and getting dollars. The solution might be written out logically as follows:

$$\frac{18760}{\underset{1000}{\cancel{2000}}} \times \$\overset{9}{\cancel{18}} = \$170.840$$

In finding 18,760×\$9, the *logical* multiplier is 18,760, but the *actual* one is 9. The denomination of the answer depends upon the denomination of the number multiplied, of course; but the real process is that of finding the product of the two abstract factors, 18,760 and 9, either of which may be used as the multiplier.

When several multiplications and divisions occur, all work should be written down, like factors canceled, and the multiplication performed before dividing. Most problems of mensuration are of this type. Thus to find the capacity of a bin 15 feet long, $8\frac{1}{2}$ feet wide, and filled to a depth of 6 feet, using .8 bu. per cubic foot, the solution is:

$$\frac{\overset{3}{\cancel{15}} \times 17 \times \overset{3}{\cancel{6}}}{\cancel{2}} \times \frac{4}{\cancel{5}} \text{ bu.} = 612 \text{ bu.}$$

After canceling like factors the multiplication can be performed without a pencil.

To find the capacity of a tank 27 inches by 42 inches by 22 inches, using 231 cubic inches per gallon, the solution is as follows:

$$\frac{\overset{9}{\cancel{27}} \times \overset{6}{\cancel{42}} \times \overset{2}{\cancel{22}}}{\cancel{3} \times \cancel{7} \times \cancel{11}} \text{ gal.} = 108 \text{ gal.}$$

Again all computation is done without a pencil. Instead of writing 231, its factors were written to facilitate cancellation.

To discount a bill of \$134.50 $33\frac{1}{3}\%$ and 10% work is

saved by discounting by 10% first, for 134.50 will not contain 3; but after taking off 10% of any number the remainder will *always* contain 3. Hence, the solution might be as given in the margin.

$$\begin{array}{r} \$134.50 \\ \underline{13.45} \\ \$121.05 \\ \underline{40.35} \\ \$80.70 \end{array}$$

The same problem, however, can be solved without a pencil by observing that, after $33\frac{1}{3}\%$ and 10% have been deducted, but 60% remains. And 60% of \$134.50 = \$80.70.

No fixed form of solution should be required. The pupil should reason out what to do and look to see if any required process will cancel any other.

Errors in Labeling Steps in the Solution

It is a common practice to require the pupil to label each step. This invariably leads to erroneous reasoning. This is clearly shown in the usual solution to the following type of problems: "Allowing 7 pk. of seed wheat to the acre, how many bushels are needed for a field 35 rd. by 42 rd.?" In many schoolrooms you will see the solution shown in the margin. It will be seen that every principle of multiplication and division has been violated. Rods cannot be multiplied by rods. Square rods divided by 160 does not give acres. Acres multiplied by 7 does not give pecks; and pecks divided by 4 does not give bushels. None of the steps should have been labeled in such a solution. The following solution

$$\begin{array}{r} 35 \text{ rd.} \\ \underline{42 \text{ rd.}} \\ 70 \phantom{\text{ rd.}} \\ \underline{140 } \phantom{\text{ rd.}} \\ 160\overline{)1470} \text{ sq. rd.} \\ 9\tfrac{3}{16} \text{ A.} \\ \underline{7 \phantom{\tfrac{3}{16}}} \phantom{\text{ A.}} \\ 4\overline{)64\tfrac{5}{16}} \text{ pk.} \\ 16\tfrac{5}{64} \text{ bu.} \end{array}$$

would not only have been correct, but it would have been much more economical:

$$\frac{\overset{7}{\cancel{35}} \times \overset{21}{\cancel{42}}}{\underset{32}{\cancel{160}}} \times \frac{7}{\underset{2}{\cancel{4}}} \text{ bu.} = \frac{1029}{64} \text{ bu.} = 16\tfrac{5}{64} \text{ bu.}$$

It is not the type of solution, however, that is important. But the important thing is that the solution of the problem is an act of the pupil's own judgment and not the application of some meaningless stereotyped rule or formula.

CHAPTER XV

PLANNING THE LESSON

Need of Preparation

Before conducting a class exercise of any sort, a teacher should make a careful preparation for the work she expects to do. She should have definitely in mind just what she is going to present and *why* and *how* she is going to present it. She should understand the whole subject so well that she knows the logical steps to follow in order to accomplish her aim; that is, she should know just what knowledge the pupils have had in order to take up the work which she has planned for the day.

Three Types of Lessons

There are three general types of lessons in the teaching of arithmetic. The work of the day may consist of developing some new fact or process; or it may be devoted to drilling upon previously developed facts or processes in order to give automatic control of them; or it may be an application of principles and processes already learned and made automatic to the solution of problems. The nature of the teacher's part in the class exercise depends upon which of these three phases is under consideration. These phases will be considered under:

(1) the drill lesson; (2) the inductive lesson; and (3) the deductive lesson.

THE DRILL LESSON

The pupil must have an automatic control of all of the fundamental facts and processes of arithmetic. This control can be accomplished only through well-chosen motivated drill. In some schools there is too great an emphasis placed upon drill. In others there is too little. The teacher should consider carefully the three phases of her work in order that they may be well balanced.

The teacher should realize, however, that accuracy and a fair degree of rapidity in the four fundamental processes are of first importance, and to secure this all other desiderata must be subordinated. To secure and keep this skill, constant practice and drill are indispensable, not only in the lower grades, when the fundamental processes are first taught, but throughout the entire course.

Need of Motivation

While drill exercises may seem mechanical and depressing, there is no substitute for them, and the ingenious teacher will seek to make them attractive through competitive tests, by working with a time limit to make the pupil conscious of his growing power, or by other forms of motivation suggested in the preceding chapters.

It is not sufficient for a child in the lower grades to be told that his future success depends upon the habits developed by this drill. He must have an immediate motive; and the more immediate the motive — the

stronger the incentive — the greater the attention that will be given to the work, and the more rapid the progress that he will make.

Drill Must Include Each Fact of a Series

Often the drill lessons are not systematically planned. They are made up at random and lack system. Easy combinations, for example, may be given more often than more difficult ones. If the drills are made up at random during the class period, many of the combinations may be repeated uselessly while others may be omitted entirely. The class period may have merely "killed time" and served as little purpose as some so-called "busy work."

When the object of the drill is to secure an automatic response to a series of facts, the teacher must see to it that every fact of the series receives the required attention. If the drill is upon the forty-five primary facts of addition, the teacher must make sure that every fact is included in the drill. This does not imply that each fact should recur exactly the same number of times. Thus, when drilling from an addition chart like the one given in Chapter IV, the latter part of the chart should receive much more emphasis than the first part of it.

Unless great care is exercised, the problems given in the lower grades for the purpose of motivating drill work are likely to make too great use of certain combinations to the exclusion of others. This is necessarily so, for the problems should represent real conditions as to prices, sizes, quantities, etc., and this limits the numbers to be used. For this reason, problems for the purpose of fur-

nishing drill work in the fundamental facts should be used but sparingly, and the teacher should exercise great care in order to get variety in the combinations involved.

As indicated above, the greater part of the time given to a drill lesson should be spent upon those facts or processes that present the greatest difficulty. Frequent testing will show the teacher which facts have come under the desired control and which ones need special drill, and her drills should be planned accordingly.

The Repetition of Drills

When a table is learned, it must be drilled upon daily until it can be accurately and quickly given. But this is not sufficient. A table may be given ever so glibly by a pupil after a few days of drill, and yet the facts that are not used for a week or so may be entirely forgotten when needed again. When tables are first learned, the periods between drills should be short. Thus, if a pupil does not have occasion to use certain facts of addition for several weeks after first learning them, he may entirely forget them. Hence, as the tables of a new process are learned, drill upon those already learned cannot be neglected. However, the frequency in which they necessarily recur gradually changes. As the child matures, the periods between drills may gradually be lengthened.

Holding the Attention

The results accomplished in any kind of drill work depend upon the degree of attention given by the pupil. How to secure a maximum of attention is the vital problem,

then, that confronts the teacher. It is difficult for children to fix their attention for a considerable length of time. A teacher could hardly expect that a maximum of attention could be given a drill that extended throughout the entire class period without variation. When the pupils show signs of lagging in attention, the form of the drill should be changed.

In concert work a large number of the class are not really attending to the drills but are saying what the leaders say and are thus getting no value from the exercise. So, while some concert drill work may occasionally be given, it should not consume an important part of the class period.

On the other hand, individual drill has to be done very skillfully, or the one reciting may be the only one paying attention to the work that is being done. It frequently becomes necessary, however, to have an individual pupil recite an entire series of facts; but there must be some incentive to get each pupil of the class silently to go through the same recitation. That was the purpose for which many of the games given in Chapter VIII were invented.

A method of securing the maximum attention of a class cannot be outlined. Much depends upon the personality and the versatility of the teacher. She cannot create interest and keep the attention of the class fixed upon the work unless she too is interested in it. A wide-awake, alert, resourceful teacher will readily devise means of securing attention as occasion for it arises; while a listless teacher, or one who lacks the power to sense a

situation, could not hold the attention of a class even though she knew all the laws upon which attention depends.

While much depends upon the resourcefulness of the teacher, among the various means of securing attention that may be listed are: (1) variety in the forms of drill; (2) working with a time limit; (3) desire to excel former records; (4) the desire of a pupil to do as well as other members of the class; (5) the desire of the class as a whole to do as well as or to excel some former class; or (6) the desire of the class to do as well as or excel a class of the same grade in some other school.

THE INDUCTIVE LESSON

Although much of the time in the primary grades is spent in drill work, still the teaching of arithmetic furnishes many opportunities to stimulate and train the pupil to analyze a situation and draw a conclusion. The traditional "development lesson," designed to stimulate thinking, is usually treated under five formal steps: (1) preparation; (2) presentation; (3) comparison; (4) generalization; and (5) application. While there is no longer a rigid adherence to these five steps followed in the order given, and without any elimination or combination, still a recognition of them will be of help whenever the inductive development of a fact or process seems desirable. These five formal steps, however, must be considered as suggestive rather than as furnishing a fixed mode of procedure.

The Preparation

To prepare a pupil to think, a problem whose solution is necessary must first be presented. Thought is stimulated only when one realizes some aim that is to be satisfied by the process. This is formally called "developing an aim."

The solution of this new problem, however, depends upon some old experience; hence, another factor in the preparation is a review of the related old experience. These two phases of the preparation: (1) the recognition of a problem whose solution is necessary, and (2) a review of the old facts upon which the solution depends, need not follow each other in this order. The second phase may even be eliminated when the teacher knows without a review that all the old facts are fresh in the pupil's mind.

Thus, before a pupil takes up the addition of two fractions whose units are unlike as $\frac{1}{3} + \frac{3}{4}$, he has met and solved the two problems in the addition of two fractions, the one whose units are alike, and the other in which one unit can be changed to the other, as for example: $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, and $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$. Now, it is not necessary to spend time in reviewing these two problems if they have immediately preceded this one, except to show wherein the new one is different from the old ones, and thus get the pupil to see the new problem before him, which is the changing of fractions to like units.

To spend time reviewing well-known facts instead of proceeding at once to the new will cause a loss of interest.

The Presentation

After a full realization of the problem to be solved, the next step is to relate the elements of the new to facts already known. No rule can be laid down that will fully cover the teacher's part in this. Sometimes, through a series of questions, she gets the pupil to relate the new with the old. At other times she finds it more economical to illustrate or demonstrate the relations and ask fewer questions. Thus, the *presentation* of the area of a circle requires many more directions and demonstrations than mere questions. After the pupil has realized that the circle cannot be divided into a number of square inches, as in the case of the rectangle, and, as it stands, it cannot be divided up into triangles, parallelograms, or trapezoids, as all plane rectilinear figures could, he has grasped the problem. At least he sees that here is an area to be measured that differs in form from the others and seems to demand a new "rule." Now in the *presentation* the teacher can show that by dividing the circle into a number of equal sectors and properly fitting them together, the circle may be converted into what is practically a rectangle or parallelogram with known dimensions. Now through a series of questions she gets from the pupil that the area must equal that of a rectangle whose base is half the circumference and whose width is the radius, which is really the next two of the formal steps — the comparison and the generalization.

Avoid questions, however, that are so suggestive that they fail to stimulate thought.

The Comparison

With the problem before us and the necessary data furnished, the pupil is led to compare the new with what is already known. Thus in the illustration given above, in finding the area of a circle, the pupil recognizes the problem and is furnished with a circle cut into sixteen or more equal sectors which are rearranged to somewhat resemble a rectangle. The pupil observes several such circles in their new arrangement and compares the forms they now assume and draws the conclusion that they are equivalent to rectangles with known dimensions.

The Generalization

After making comparisons and forming conclusions until satisfied of certain fixed facts that hold good in all cases, the pupil summarizes these facts into a definition or principle, or into a working rule for later use. Thus, in the example given, the pupil generalizes his conclusions and states that the area of a circle is equal to that of a rectangle whose base is half the circumference and whose width is the radius. Or, stated in the form of a rule, to find the number of units in the area of a circle, multiply the number of units in half the circumference by the number in the radius.

Through the aid of the teacher, the rule is then fitted into the conventional form to be memorized, as:

$$A = \pi \times R^2 = \frac{\pi}{4} \times D^2$$

Besides aiding directly in forming habits of right thinking, the discovery and generalization of facts and processes by the pupil for himself give him a better grasp of the subject so that it is used more intelligently and remembered much more easily. The mere ability to repeat parrot-like the rules or definitions of arithmetic with no concrete background as a basis for them is practically valueless.

The Application

When conclusions have been reached and generalizations made, opportunity for them to be used in some practical problem must be furnished. In so far as possible, the applications found in the textbook must be supplemented by problems that some one in the life about them needs to solve.

Some Cautions in Using the Inductive Method

A teacher in planning the development of a rule or principle must not feel that in order to do it properly the five formal steps must all be followed as five distinct, exclusive steps. Often there is no sharp differentiation between them. In seeking to bring out the five steps, she may make the work formal and uninteresting. Long, tedious presentations should be avoided. If a truth has been grasped and can be applied, that is the supreme test.

Do not give too much help. Some teachers interpret the inductive method to mean that, through a series of carefully planned questions, the pupil is brought to state

the generalization. Often these questions are so suggestive as to require no thought upon the part of the pupil. After the problem is fully presented to the pupil, the more he does for himself, the better. It is the teacher's part to stimulate the pupil to seek the solution, but to give as few hints and to ask as few leading questions as possible.

The Formal Steps in Teaching the Area of a Rectangle

This is taken up here to illustrate the five formal steps of an inductive lesson that have just been discussed. By referring to the method of developing the various topics given in former chapters the teacher should be able to thus arrange the steps in the teaching of any topic.

Preparation. — Recall the methods of measuring other magnitudes, as lines, the contents of vessels, etc. See that the pupil understands that to measure any magnitude is to find how many of some standard unit it will contain. Show some standard units used in measuring surfaces, as a square inch, a square foot, etc. Show some rectangle whose dimensions are integers and measure it by placing enough square units on it to cover it. Raise the question of how the number needed to cover it could have been calculated so as to save the necessity of covering the area with square units. The problem is thus fully presented; that is, the teacher has developed "the pupil's aim."

Presentation. — Furnish pupils with rectangles of cardboard and with but one square unit each with which to measure them. Let them find the area in any way they can. Encourage them to thus lay off the rectangles into

square units. Ask them to compare the number of square units in one row along the length with the number of linear units in the length. Then ask them to compare the number of such rows with the number of linear units in the width.

Comparison. — Have the pupils make several comparisons as above and draw conclusions until they can tell without drawing that a rectangle 3 in. by 4 in. is composed of 3 rows of 4 sq. in. each, that one 5 in. by 6 in. is composed of 5 rows of 6 sq. in. each, etc. From this have them see that in a rectangle 3 in. by 4 in. there are 3×4 sq. in. In one 5 in. by 6 in. there are 5×6 sq. in., etc.

Generalization. — From the conclusions formed above, have the pupils formulate the fact that the number of square units in the area of a rectangle is the product of the number of linear units in the two dimensions.

Application. — Test the pupils' understanding of the principle just stated by applying it to problems of the textbook, then to those found out of the text. In measuring rectangles, select those whose measurement is necessary. Thus, how much did the walk in front of the school cost? Get the local price per unit and thus answer the question. There is an abundance of real problems on every hand that require the measurement of rectangles.

The Lesson Plan

In writing out a lesson plan embracing the five steps as given above, it is customary to ask questions to show the "teacher's part" and then answer them to show the

"pupil's part." While a teacher should have clearly in mind just the questions that she intends to ask, she must realize that the pupils will not always answer as she has expected them to do and hence that her questions will have to be changed. Aside from the general questions, then, she must fully realize what each step consists of and how she expects to attain the ends sought and then secure the greatest possible self-activity from the pupil. Questions are often so leading that the pupil has to do but little, if any, thinking. This should be carefully avoided. The supreme test of teaching is the ability of the teacher to arouse the self-activity of the pupil.

THE DEDUCTIVE LESSON

The complete thought process involves both the inductive and deductive types of thinking. A lesson is rarely ever purely inductive or purely deductive. All laws are first arrived at through the process of induction; but, in verifying them, the thought is deductive. The present plan of developing the principles of arithmetic is more inductive than deductive. The applications of these principles are more deductive. However, in teaching a process or rule, the plan sometimes becomes more deductive than inductive. Thus, in teaching the rule for finding the area of a rectangle given under the inductive plan, had the teacher equipped the class with rectangles ruled in squares or with squared paper and then asked them to verify the fact that the number of square units was equal to the product of the number of linear units

in the two dimensions, the process would have been deductive. Whether the teaching process is to be inductive or deductive, the first step is the *preparation;* that is, needed knowledge is reviewed and the problem fully presented. A *generalization* is then assumed, and, by appealing to known facts, it is *verified* through a course of reasoning.

In the solution of a problem, the type of reasoning is largely deductive. Thus, there is first a clear recognition of the problem. Then follows an analysis of conditions in seeking the principles to be applied. In general, it is more difficult to select and classify the formal steps, particularly in the solution of a problem, than it was in the inductive type. Thus, if one is to find a selling price of an article costing \$2.10 that will give a profit of 30% of itself, the problem is first studied, and it is clearly seen that this problem is one in which the basis upon which the per cent is reckoned is not given, but is wanted. Hence, it cannot be the first type of problem. Then one reasons that, since 30% of the selling price must be profit, the remainder of it, or 70% of it, must represent the cost. Hence, the relation 70% *of the selling price equals* \$2.10 is seen. Now, if this type of relation is a seemingly new one, the reasoning again is somewhat as follows: This means .70 × an unknown selling price = \$2.10. But this means that the product of two factors and one of the factors is known and the problem is to find the other. But from the meaning of division the solution must be \$2.10 ÷ .70. Hence, the selling price must be \$3. To verify this, the gain must be 30% of

$3 or 90 cents. And this leaves $2.10 for the cost as it should. Hence, the solution is correct. Thus, it is seen that there is a series of problems arising, each followed by an analysis and a conclusion of which principles to apply, and finally a verification of the conclusion.

CHAPTER XVI

THE COURSE OF STUDY IN ARITHMETIC

One of the chief factors in efficient school work is the course of study. And, while it must not be inflexible, yet it must be based upon certain fixed fundamental principles. Each year's work must be based as nearly as possible upon the needs, interests, experiences, and personality of the learner, and made to function with his life both in and out of school. That is, the material used must be based upon some social issue of present or near future value and not given for the mere purpose of training.

There are three natural divisions of the course in arithmetic, each with its own aims, methods, and material all rather distinctly marked. These three divisions fall naturally under the so-called Primary, Intermediate, and Grammar School grades. Textbook makers have followed this division in their texts, and most modern textbooks thus become a sort of course of study. But the matter of any textbook must be adapted to the ends to be attained by omitting and supplementing to meet the needs of the child, for (it must not be forgotten that it is the child and not the subject that is being developed.) It is his temperament, his interests, his needs, and his abilities that must be studied, and the instruction must be shaped to fit them.

THE PRIMARY GRADES

The primary grades include the work of the first four years. The aims of these grades are clearly defined and definite. They are: (1) to teach the reading and writing of numbers; (2) to give an automatic control of the primary number facts (the so-called tables); and (3) to develop some skill in the written processes with whole numbers. While, of course, all these will find applications within the child's interests and needs, the problems of the primary grades are of minor importance except in so far as they make clear the meaning of the processes and motivate the drill work. Neither is the rationalization of the fundamental processes with whole numbers of importance. It is proper habits of procedure that are of chief importance. Thus the chief problem of the teacher of these grades is how to bring about a control of the primary facts and dependable habits of procedure in the written processes with the greatest economy of time and effort. The problem, then, is a study of methods of drill and of the proper gradation of the written work — a study of the means of vitalizing the work so as to secure a maximum of attention. These have all been discussed in former chapters of this book.

The First Grade

Following the principle that the work of each year shall as nearly as possible be based upon the child's needs and interests, there is a rapidly growing tendency to defer drill in formal number work until the second school year.

However, there are needs of *counting* and of *expression* during the first year that should be met. In most localities the child's needs during the first year are limited to (1) *reading numbers to* 100, or possibly to reading numbers of three figures, in order to find a page in his book or the number on a house, if he lives in a city; (2) *reading the Roman numerals to XII on a clock face,* in order to tell time; (3) *counting by ones and tens to* 100, in order to count the words or lines in a lesson or the objects with which he works in other school activities; (4) possibly the recognition of *one half, one third, and one fourth* of an object that has been divided; and also (5) familiarity with the common units of measure used by him in or out of school. All this may be done in connection with other school work, without setting aside a period in the program especially for number work.

The Second Grade

The work of this grade should be entirely oral. The most important aims of the year's work are: (1) to make automatic the forty-five primary number facts of addition and the corresponding eighty-one subtraction facts; (2) to develop some skill in calling the sum of any two-figured number added to a one-figured number; and (3) to develop some ability in adding three or four one-figured numbers. Compared with these three aims, the other arithmetic work of the year is of minor importance. At the end of the second year, then, there should have been accomplished the following:

1. Reading and writing numbers to 1000.

2. Reading Roman numerals to XII.

3. Telling time to hours and half hours.

4. Reading and writing dollars and cents.

5. Interpreting the signs +, −, =, \$, ¢.

6. The forty-five primary facts of addition and the corresponding eighty-one subtraction facts.

7. Some skill in adding at sight any one-figured number to any two-figured number.

8. Some skill in adding single columns of three or four one-figured numbers.

9. Recognition of halves, thirds, and fourths of single objects.

10. Those measuring units used in any of the child's activities either in or out of school.

Suggestions. — 1. Careful instruction and much practice in making the figures and signs should be given, so that in later written work the making of these will not retard thinking. While all the work of this grade is *oral*, the pupil should spend much time in copying figures and his "tables" just as he copies letters and words.

2. Addition and subtraction should be treated as correlative processes. Thus, when a child learns that 3 and 5 are 8, he learns that either part taken from 8 leaves the other.

3. Drill with charts and flash cards until all the combinations can be given instantly at sight.

4. Observe from Chapter IV that the forty-five primary number facts of addition are more economically treated by dividing them into *two groups:* (1) those twenty-five combinations whose sums do not exceed 10; and (2) those twenty whose sums do exceed 10. The sums of the first group are found through counting in order to fix the meaning of addition. The sums of the second group are not found through counting. A child is not able to image eight or nine things, much less their sum. No aid, then, to memorizing the second group is secured

through finding the sums through counting, and the finding of the first group is sufficient to fix the meaning of addition. (See Ch. IV.)

5. The drills should be written in column form as the child is to see them later in written work. Sight work should precede dictation work, but both forms of drill should be used.

6. Make much use of games to vitalize the work and secure the maximum of attention.

The Third Grade

The pupil should have a textbook in this grade and written work begins here. The most important aims of this year are: (1) to develop some skill in written addition and subtraction; (2) to make automatic the primary number facts of multiplication and the corresponding division facts; and (3) to teach written multiplication and division by one-figured multipliers and divisors, the dividend being limited to an exact multiple of the divisor. Compared with these, the other aims of the year are of minor importance. In all, the work of the year should include:

1. Reading and writing numbers to 10,000.
2. Reading and writing Roman numerals to XII.
3. Reading and writing unit fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, to $\frac{1}{9}$.
4. Review of the primary facts of addition and subtraction for accuracy and rapidity.
5. More practice in adding one-figured numbers to two-figured numbers.
6. More practice in adding at sight single columns of three or four one-figured numbers and extended to include five or six one-figured numbers.
7. Written addition limited to five or six addends.

8. Written subtraction.

9. The primary facts of multiplication and the corresponding division facts.

10. Written multiplication and division when the multipliers and divisors are one-figured numbers.

SUGGESTIONS. — 1. Facility in reading and writing numbers of three or four orders should come largely incidentally from the ordinary use of numbers, not from teaching and drilling upon "notation and numeration" as separate topics.

2. The fractions arise in the partition phase of division and should be confined to unit fractions.

3. Drill upon the primary facts of the second grade should be continued until all are recognized instantly.

4. Children should not be given much column addition until they can add two-figured and one-figured numbers with considerable skill.

5. Pupils need not understand the "why" of carrying or borrowing in this grade. The "how" is the important thing.

6. Pupils should make their multiplication tables through adding equal addends in order to see what multiplication means.

7. Include any unit of measure needed by the pupil to interpret his other work, using the unit itself as it is being taught.

8. All applications should be kept within the range of the pupil's experiences and interests. Dramatized activities as playing store, etc., are much better for this grade than problems from the world of adults.

THE FOURTH GRADE

The greater part of the time of this year is devoted to securing greater skill in the work begun in the third year. The *new* work of the year consists chiefly in long multiplication and long division, using multipliers and divisors of two or three figures. At the end of the fourth grade the work finished should include:

1. Reading and writing numbers to 100,000,000.

2. Reading and writing any of the Roman numerals.

3. Skill in written addition and subtraction and power to check results so as to turn in work 100% accurate.

4. Complete mastery of the multiplication and division tables.

5. Skill in written multiplication and division where multipliers and divisors do not exceed three-figured numbers, and power to check work in order to turn in work that is 100% accurate.

6. Power to apply the knowledge thus far acquired to problems that come within the pupils' needs and experience.

Suggestions. — 1. Do not perplex pupils with drill in "notation." The reading of numbers is of more importance. Facility in reading and writing numbers comes incidentally through the ordinary use of numbers rather than through special drill.

2. The reading and writing of the Roman numerals is of minor importance and should not consume time needed for more important parts of the year's work.

3. All drills of the preceding grades should be continued to give greater skill.

4. Confine the work of denominate numbers to the needs of the pupil. When a new unit is introduced it should be shown the class. If square and cubic units are taken up, see that pupils get a clear conception of them and their use.

THE INTERMEDIATE GRADES

This includes the work of the fifth and sixth grades. The teaching in these grades takes on new phases. Rationalization plays a much more important part. The processes with fractions and decimals recur less frequently

than those with whole numbers, hence rationalization is needed in order to help the memory retain the "how" of the processes. Not only on that account, but because the rationalization of the subject of fractions and decimals develops greater power to use the subjects properly when applied to problems is rationalization urged.

The applications of arithmetic to problems becomes much more important in these grades than in the primary grades. The problems are not given now merely to furnish occasions for calculation, but they are connected more with some social issue of interest to the pupil and thus serve to develop the habit of looking upon the quantitative side of life. He begins, too, in these grades to get some insight into the use of arithmetic in commercial and industrial phases of the world's work.

The Fifth Grade

Drill in the work of the preceding grades should continue in order to secure greater skill. Greater use of the work of whole numbers in problems should be made. But the *new* work of this year is chiefly the development of the fundamental processes with fractions and mixed numbers and application of these processes to problems that need solution. The complete work of the year should include the following:

1. Review and extend the work in whole numbers.
2. Develop multiplication where the multiplier contains one or more zeros.
3. Take up division in which zeros occur in the quotient.
4. Teach the meaning of abstract and concrete numbers.

5. Teach all four fundamental processes with fractions and mixed numbers.

6. Teach the meaning of ratio and use ratio within the limits of familiar numbers.

7. Teach the measurement of areas of rectangles, parallelograms, and triangles.

8. Teach the surfaces and volumes of right prisms.

SUGGESTIONS. — 1. In the applications do not allow a pupil to speak of the concrete factor as the multiplier.

2. Develop the notion of a fraction objectively.

3. Be careful in developing the notation of a fraction that the pupil sees the use of each term. The development of the processes depends upon the notation.

4. Show that as long as the meaning of a process does not change, the method does not change.

5. Relate all processes in fractions with their corresponding processes with integers.

6. Have pupils see clearly that the so-called multiplication by a fraction is a new meaning of multiplication. $\frac{3}{4} \times \frac{5}{6}$ does not mean "$\frac{3}{4}$ times $\frac{5}{6}$" but "$\frac{3}{4}$ of $\frac{5}{6}$."

7. Seek local applications for all processes as fast as learned.

THE SIXTH GRADE

Fractions, whole numbers, and mensuration are taken up in this grade for review and extended in their applications. The *new* work of this grade consists of decimals and percentage. These, then, demand the greatest care in presentation. The outline of the year's work is as follows:

1. Teach the decimal place-value feature of our notation.

2. Continue drill in whole numbers, using larger numbers.

3. Apply addition and subtraction to keeping accounts.

4. Teach short methods of multiplication and division, using aliquot parts and special fractions.

5. Review and extend the meaning and use of common fractions.

6. Develop cancellation.

7. Develop the meaning of ratio and how to express a ratio as a fraction.

8. Teach the fundamental processes with decimals.

9. Express any ratio in decimal form.

10. Develop per cent as a new name and notation for a special fraction.

11. Study the application of percentage, confining the work to the *two* direct applications of the subject: (1) finding a per cent of a number; and (2) finding what per cent one number is of another.

12. Review the mensuration given in the fifth grade and extend its applications.

SUGGESTIONS. — 1. In this final study of fractions seek to develop the full meaning and use of a fraction: (1) one or more of the several parts of some whole; (2) an expressed division; and (3) the expression of the ratio of one magnitude to another.

2. Develop the decimal fraction as an extension of our decimal place-value notation to the right of ones' place, not as a special notation of a common fraction.

3. It is very important that a pupil be able to express the ratio, as a common fraction, between any two magnitudes, then change this ratio to a decimal, for this is the foundation of one of the most important problems of percentage.

4. Present percentage as a problem of decimals, the notation being the only new feature.

5. But two problems of percentage should be taught this year — to find a per cent of a number, and to find what per cent

one number is of another. Make clear that both of these problems were encountered in decimals, — multiplying by a decimal, and expressing a ratio as a decimal.

6. Nothing is gained in clearness of thought by using the terms "base," "rate," and "percentage," terms still used in some textbooks. Hence it is better not to use them except when "rate" is used in such expressions as "rate of gain," "rate of interest," "rate of discount," etc.

7. While the applications now begin to take on more and more of the adult point of view, keep them well within the pupil's experiences. Encourage pupils to bring in advertisements, sample business papers, bills, accounts, notes, etc. Make the commercial applications more real and concrete by dramatizing the adult activities that are studied. Children like to play "Going into business."

8. In local applications of mensuration have the pupils make their own measurements in getting the needed data.

THE GRAMMAR SCHOOL GRADES

This includes the work of the seventh and eighth grades. Educational opinion seems less settled upon what should be included in these two grades than in the earlier grades. In most schools the work still consists almost entirely of arithmetic. In schools organized under the Junior High School plan some algebra and observational geometry are being introduced in these grades and arithmetic continued into the ninth grade.

It seems clear, however, that in so far as the maturity of the pupil will permit, the work of these two grades should furnish the mathematics needed by the average intelligent citizen outside of a specialized vocational need. The needs of such a person are (1) *power* to see and express, and to interpret the expressions of the quantitative relations that come within one's needs and

interests; (2) the *habit* of looking upon the quantitative side of life and seeing these relations, particularly those vital to one's welfare; and (3) a *knowledge* of commercial and industrial practices through which one may interpret references met in general reading and in one's business and social intercourse with those with whom he comes in contact.

With these aims, then, as the basis of the course, the subject is organized around some social topic instead of around some arithmetical topic, as borrowing and loaning money, buying stocks, bond investments, taxes, insurance, etc. Or, these are sometimes included as parts of larger units, as problems of thrift, problems of investment, problems of protection, problems of public expense, problems of transportation, problems of industrial life, etc. The algebra taught in these grades is usually limited to the interpretation of the formula and to the use of the simple equation. The mensuration of the past is now sometimes supplemented by a little constructive and observational geometry and listed under the general topic of "geometry." Regardless of the grouping, the following topics are usually included in these two grades:

The Seventh Grade

1. A review of whole numbers, fractions, and decimals, including short methods and emphasis upon checking results and estimating results.

2. The two problems of percentage that were taken up in the sixth grade as applied to profit and loss; discount; commission; simple interest; bank discount.

3. The measurement of the areas of rectangles, triangles, parallelograms, trapezoids, and circles; and the measurement of the surfaces and volumes of prisms and cylinders as applied to practical problems.

4. Drawing to a scale and sketching and interpreting plans and diagrams.

SUGGESTIONS. — 1. Attention this year to the applications of arithmetic is the important thing. These applications include denominate numbers, mensuration, and percentage.

2. Confine the work of percentage to the two practical problems of the sixth-grade outline. Fix in memory all the important fractional equivalents by constant use and drill.

3. Seek to make all work applied to commercial transactions as realistic as possible. Play "Going into business" if it adds interest and makes the work more realistic. All commercial schools do this with students of much greater maturity.

4. Encourage pupils to bring in real problems, commercial papers, and any material that will vitalize the work.

5. Distinguish between *real problems* met in real life, and problems "for analysis" about *real* things. Remember that a "real" problem is not necessarily concrete to the pupil.

6. In the application of mensuration, encourage pupils to bring in real problems from measurements which they have made for themselves.

7. Problems in mensuration are usually made clearer by diagrams or drawings made to a scale.

THE EIGHTH GRADE

The work of the eighth grade is an extension of the work of the seventh to a wider range of applications and it takes on a more adult point of view. If square root is to be taught at all, it is taken up in this grade. If the work of arithmetic is given in order that the pupil may have the ability to interpret the quantitative phases

of his environment, it would seem that the time spent in square root and its applications might better be spent with other problems.

At the end of this year fundamental principles should be thoroughly understood, habits of accuracy fixed, and readiness and a fair degree of speed in ordinary computations attained. Power to state a problem clearly, to analyze it logically, to choose a good method of solution, and to do the work by the shortest method should have been acquired. The work, then, should include:

1. A general application of the work of the preceding grades to problems of human interest.

2. An extension of percentage to insurance; taxes; national revenues; trade discount; successive trade discounts; simple interest; bank discount; stock investments; bond investments.

3. An extension of mensuration to pyramids, cones, and the Pythagorean Theorem. Also some simple geometric constructions and observations; and the three forms of graphs used in representing statistics.

4. The use and interpretation of formulæ as "short-hand expressions" of the principles and rules of mensuration.

5. The use of ratio and proportion in expressing the relation of similar figures and in the problems of simple machines, as the lever, inclined plane, screw, wedge, etc.

Suggestions. — 1. If time permits, the indirect problem of percentage may be taken up. There are but few real applications of it. It is needed to find a selling price that will yield a certain per cent of profit upon itself when the cost is known;

also to find a list price that may be discounted and yet yield a certain profit when the cost is known.

2. The applications of percentage are difficult to teach only because the transactions are outside of the experiences of many of the pupils. Seek to make the work as realistic as possible. Pupils of this age usually like to play at "make-believe" billing clerks, retail merchants, wholesalers, bank cashiers, etc. This is done in all commercial schools.

3. Encourage pupils to bring in all kinds of commercial papers — price lists, invoices of goods, receipts, checks, promissory notes, insurance policies, tax bills, etc. As these are collected by the pupils they should be carefully preserved and kept "in stock" as part of the illustrative material of the school.

4. In "stock investments" there are but two *real* problems, viz.: (1) Stock bought at one price and sold at another is a gain or a loss to the buyer of how much; (2) stock costing a certain price and paying a certain dividend makes the investor what per cent of the investment. These problems are no more difficult than many encountered much earlier in the course. The pupil's difficulty with them lies in the fact that they are not concrete to him. They lie entirely without his range of experiences. Seek to make the topic realistic by a study of some local corporation, by the issuing of stock certificates upon "make-believe" corporations organized by the pupils, and by use of problems made by the pupils from the market reports of the daily papers.

5. The study of bonds may be made more real in connection with the study of civil government in considering the methods used by municipalities, states, or governments of raising money. It is best to begin with a study of your own city.

6. It is well to explain the functions of savings banks, banks of deposit, and other corporations. A visit to such places is well worth while.

CHAPTER XVII

MEASURING RESULTS

UNTIL recently the rating of a pupil has depended upon the personal opinion of the teacher and upon set examinations. During the last few years considerable work has been done along a method of measuring results, known as "standard tests," that promises a more scientific means of measuring the products of certain lines of school training.

USES OF STANDARDS IN MEASURING EFFICIENCY

There are at least three general uses or purposes of standards in measuring efficiency. There is a need of tests and standards to be used in the administration of a school system, also, for determining the rating of an individual for purposes of promotion; and, finally, tests for the purpose of analyzing the primary defects that lead to backwardness in any subject. These may be classified as administrative, grading, and diagnostic tests. A standard test that will meet one of these three purposes need not necessarily meet another. Thus, from the standpoint of administration, the test must show merely the general ranking of a large group, or of an entire school system in a finished product; as, for example, written addition. But the ability to do written addition is a complex

consisting of several primary abilities with which the administration is not directly concerned.

From the standpoint of grading a pupil for entrance or promotion, the test must measure the individual ability instead of group ability, and, hence, must be so made as to eliminate more nearly the element of chance error. Thus the average score from a single example in addition given to several thousand pupils might mark fairly well the general ability of that group in that subject and thus answer the purposes of an administrative test; but such a test would be entirely unreliable for a single individual, owing to the element of chance. A grading test, moreover, must be a test of the ability of the individual in a finished product, not of abilities needed in that finished product.

The diagnostic tests are tests given to discover causes of backwardness in performing any finished process. They must locate the particular primary ability or abilities that need special training. The weakness in any final process may be due to a weakness in one or more of several practically unrelated primary abilities. Thus failure in the finished product of written addition may be due to the lack of any one of the following primary abilities: (1) the automatic control of the primary number facts; (2) ability to hold in mind a two- or a three-figured number and add it to a one-figured number; (3) ability to carry in mind the "tens" of each column-sum and add them to the next column; and (4) ability to record accurately the "ones" figure of each column-sum. Ability in any one of these does not imply ability in any other, so diagnostic tests to show the source of weakness must be made

before corrective drill work can be wisely applied. Thus, if the weakness in final addition is due to (2), outlined above, as is often the case, drill upon (1) is of almost no help, yet this is often the only source of training preparatory to written work. Thus it is seen that tests to be of any real service in supervising, promoting, or correcting weaknesses must be worked out carefully according to the purposes that they are to serve.

Attempted Solutions of the Problem

While many persons have worked at the problems outlined above, usually the work has been restricted to special classes or school systems. Of those who have worked in a wider field, the work of Rice, Stone, Courtis, and Woody is best known. The real problem of Rice and Stone, however, was not that of establishing a standard of measurement, but since their questions and results are used by school systems in comparing their own schools with those examined by these men, their questions are often classed among the "standard tests."

The Work of Rice

The tests given by Mr. J. M. Rice[1] in 1902 were not intended to become standards. They consisted of five sets of questions of eight each, for grades four to eight, inclusive, and were given to the pupils of seven city systems, aggregating over 6000 pupils, in order to determine some of the factors upon which successful school work depends. They are of interest in showing the beginning

[1] *The Forum*, XXXIV (October–December), 1902.

of the attempts that later led to more scientific attempts at standardization.

The Questions Used by Rice

Fourth Year

1. A man bought a lot of land for $1743, and built upon it a house costing $5482. He sold them both for $10,000. How much money did he make?

2. If a boy pays $2.83 for a hundred papers, and sells them at 4 cents apiece, how much money does he make?

3. If there were 4839 classrooms in New York City, and 47 children in each classroom, how many children would there be in the New York schools?

4. A man bought a farm for $16,575, paying $85 an acre. How many acres were there in the farm?

5. What will 24 quarts of cream cost at $1.20 a gallon?

6. A lady bought 4 pounds of coffee at 27 cents a pound, 16 pounds of flour at 4 cents a pound, 15 pounds of sugar at 6 cents a pound, and a basket of peaches for 95 cents. She handed the storekeeper a $10 note. How much change did she receive?

7. I have $9786. How much more must I have in order to be able to pay for a farm worth $17,225?

8. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much money do I make?

Fifth Year

1. A man bought a lot of land for $1743, and built upon it a house costing $5482. He sold them both together for $10,000. How much did he make?

2. If a boy pays $2.83 for a hundred papers, and sells them at 4 cents apiece, how much does he make?

3. What will 24 quarts of cream cost at $1.20 a gallon?

4. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much money do I make?

5. A flour merchant bought 1437 barrels of flour at $7 a barrel. He sold 900 of these barrels at $9 a barrel, and the remainder at $6 a barrel. How much did he make?

6. How many feet long is a telegraph wire extending from New York to New Haven, a distance of 74 miles? There are 5280 feet in a mile.

7. A merchant bought 15 pieces of cloth, each containing 62 yards. He sold 234 yards. How many dress patterns of 12 yards each did he have left?

8. Frank had \$3.08. He spent $\frac{1}{4}$ of it for a cap, $\frac{1}{7}$ of it for a ball, and with the remainder bought a book. How much did the book cost?

Sixth Year

1. If a boy pays \$2.83 for a hundred papers, and sells them at 4 cents apiece, how much does he make?

2. What will 24 quarts of cream cost at \$1.20 a gallon?

3. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much do I make?

4. A flour merchant bought 1437 barrels of flour at \$7 a barrel. He sold 900 of these barrels at \$9 a barrel, and the remainder at \$6 a barrel. How much did he make?

5. If a train runs $31\frac{2}{3}$ miles an hour, how long will it take the train to run from Buffalo to Omaha, a distance of 1045 miles?

6. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

7. The salt water which was obtained from the bottom of a mine of rock salt contained 0.08 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

8. A gentleman gave away $\frac{1}{7}$ of the books in his library, lent $\frac{1}{6}$ of the remainder, and sold $\frac{1}{5}$ of what was left. He then had 420 books remaining. How many had he at first?

Seventh Year

1. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

2. The salt water which was obtained from the bottom of a mine of rock salt contained 0.08 of its weight of pure salt. What

weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

3. A gentleman gave away $\frac{1}{7}$ of the books in his library, lent $\frac{1}{6}$ of the remainder, and sold $\frac{1}{5}$ of what was left. He then had 420 books remaining. How many had he at first?

4. A farmer's wife bought 2.75 yards of table linen at $0.87 a yard and 16 yards of flannel at $0.55 a yard. She paid in butter at $0.27 a pound. How many pounds of butter was she obliged to give?

5. If coffee sold at 33 cents a pound gives a profit of 10 per cent, what per cent of profit would there be if it were sold at 36 cents a pound?

6. Sold steel at $27.60 a ton, with a profit of 15 per cent, and a total profit of $184.50. What quantity was sold?

7. If a woman can weave 1 inch of rag carpet a yard wide in 4 minutes, how many hours will she be obliged to work in order to weave the carpet for a room 24 feet long and 24 feet wide? No deduction to be made for waste.

8. A fruit dealer bought 300 apples at the rate of 5 for a cent, and 300 at 4 for a cent. He sold them all at the rate of 8 for 5 cents. What per cent did he gain on his investment?

Eighth Year

1. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

2. The salt water which was obtained from the bottom of a mine of rock salt contained 0.08 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

3. A gentleman gave away $\frac{1}{7}$ of the books in his library, lent $\frac{1}{6}$ of the remainder, and sold $\frac{1}{5}$ of what was left. He then had 420 books remaining. How many had he at first?

4. A man sold 50 horses at $126.00 each. On one half of them he made 20 per cent, and on the other half he lost 10 per cent. How much did he gain?

5. Sold steel at $27.60 a ton, with a profit of 15 per cent, and a total profit of $184.50. What quantity was sold?

6. A fruit dealer bought 300 apples at the rate of 5 for a cent,

and 300 at 4 for a cent. He sold them all at the rate of 8 for 5 cents. What per cent did he gain on his investment?

7. The insurance on $\frac{2}{3}$ of the value of a hotel and furniture cost $420.00. The rate being 70 cents on $100.00, what was the value of the property?

8. Gunpowder is composed of niter 15 parts, charcoal 3 parts, and sulphur 2 parts. How much of each in 360 pounds of powder?

Results Found by Rice

The results of the tests and many interesting conclusions derived from them can be found in an article by Mr. Rice in *The Forum* for October–December, 1902. The following chart shows the averages for the schools in each of the seven cities.

City	Grade IV	Grade V	Grade VI	Grade VII	Grade VIII	School Average
III	68.4	79.5	79.3	81.1	91.7	80.0
I	72.7	84.7	80.4	64.2	80.9	76.6
I	—	80.3	80.9	43.5	72.7	69.3
I	54.5	74.7	72.2	63.5	74.5	67.8
I	60.0	70.8	69.6	54.6	66.5	64.3
II	81.3	78.2	71.2	33.6	36.8	60.2
III	70.1	53.6	43.7	53.9	51.1	54.5
IV	70.5	73.2	58.9	31.2	41.6	55.1
IV	62.9	70.5	59.8	—	22.5	53.9
IV	53.5	53.5	42.3	16.1	48.7	42.8
IV	59.8	65.3	54.9	35.2	43.5	51.5
V	38.5	67.0	44.1	29.2	51.1	45.9
VI	28.1	38.1	68.3	33.5	26.9	39.0
VI	41.6	45.3	46.1	19.5	30.2	36.5
VI	36.8	55.0	34.5	30.5	23.3	36.0
VII	59.3	53.7	35.2	29.1	25.1	40.5
VII	47.4	65.4	35.2	15.0	19.6	36.5
VII	41.1	37.5	27.6	8.9	11.3	25.3
Gen. Av.	59.5	69.4	60.7	39.4	49.4	55.7

The Work of Stone

Dr. C. W. Stone,[1] in 1908, made a study of "Arithmetical Abilities and Some of the Factors Determining Them." This was a study of the nature of the product of the first six years' work in arithmetic. The material gathered was over 6000 test papers from 152 classrooms in 26 different school systems. The questions and results have been so widely used that they are sometimes known as "Stone's Standard Tests." A study of them will show that they do not meet the needs of a standard test for any of the three uses outlined in the first part of this chapter, nor were they so intended by Dr. Stone. They have their value, however, and are reproduced here on account of the general interest in them. Pupils were allowed 12 minutes for the test on fundamental operations and 15 minutes for the test on reasoning.

The Stone Tests

Test in Fundamental Operations

Work as many of these problems as you have time for; work them in order as numbered.

1.	Add	2375
		4052
		6354
		260
		5041
		1543

[1] C. W. Stone, *Arithmetical Abilities and Some Factors Determining Them*, Published by Teachers College, Columbia University, New York City, 1908.

2. Multiply 3265 by 20.

3. Divide 3328 by 64.

4. Add

596
428
94
75
302
645
984
897

5. Multiply 768 by 604.

6. Divide 1918962 by 543.

7. Add

4695
872
7948
6786
567
858
9447
7499

8. Multiply 976 by 87.

9. Divide 2782542 by 679.

10. Multiply 5489 by 9876.

11. Divide 5099941 by 749.

12. Multiply 876 by 79.

13. Divide 62693256 by 859.

14. Multiply 96879 by 896.

Test in Reasoning

Solve as many of the following problems as you have time for; work them in order as numbered:

1. If you buy 2 tablets at 7 cents each and a book for 65 cents, how much change should you receive from a two-dollar bill?

2. John sold 4 Saturday Evening Posts at 5 cents each. He kept $\frac{1}{2}$ the money and with the other $\frac{1}{2}$ he bought Sunday papers at 2 cents each. How many did he buy?

3. If James had 4 times as much money as George, he would have \$16. How much money has George?

4. How many pencils can you buy for 50 cents at the rate of 2 for 5 cents?

5. The uniforms for a baseball nine cost \$2.50 each. The shoes cost \$2 a pair. What was the total cost of uniforms and shoes for the nine?

6. In the schools of a certain city there are 2,200 pupils; $\frac{1}{2}$ are in the primary grades, $\frac{1}{4}$ in the grammar grades, $\frac{1}{8}$ in the High School and the rest in the night school. How many pupils are there in the night school?

7. If $3\frac{1}{2}$ tons of coal cost \$21, what will $5\frac{1}{2}$ tons cost?

8. A news dealer bought some magazines for \$1. He sold them for \$1.20, gaining 5 cents on each magazine. How many magazines were there?

9. A girl spent $\frac{1}{8}$ of her money for car fare, and three times as much for clothes. Half of what she had left was 80 cents. How much money did she have at first?

10. Two girls receive \$2.10 for making button-holes. One makes 42, the other 28. How shall they divide the money?

11. Mr. Brown paid one third of the cost of a building; Mr. Johnson paid $\frac{1}{2}$ the cost. Mr. Johnson received \$500 more annual rent than Mr. Brown. How much did each receive?

12. A freight train left Albany for New York at 6 o'clock. An express left on the same track at 8 o'clock. It went at the rate of 40 miles an hour. At what time of day will it overtake the freight train if the freight train stops after it has gone 56 miles?

Percentage of Mistakes

Stone published 38 tables from the scores obtained. Those of most interest to teachers and supervisors of arithmetic are Tables VIII and IX, showing errors in reasoning and in addition.

TABLE VIII

Mistakes in Reasoning			
Systems in order of per cent of mistakes	Per cent incorrect	No. of problems incorrect	No. of problems attempted
XVI	45.1	359	796
XVII	44.9	335	746
XXIII	41.1	238	579
III	36.7	282	769
I	34.7	269	776
XV	33.7	249	739
VI	31.8	233	733
XXII	30.9	196	634
X	29.7	232	781
XXIV	28.9	167	577
XIII	28.8	230	799
XXVI	28.6	276	964
VIII	27.9	192	689
XVIII	27	175	648
XIX	26.4	255	965
XXV	25.3	150	592
IV	25.1	147	585
XX	23.4	139	595
VII	23.1	189	819
XIV	22.9	175	765
XII	22.3	185	831
IX	20.3	161	794
II	19.7	137	696
V	18.6	171	919
XXI	15.7	106	674
XI	14.4	112	776

TABLE IX

Mistakes in Addition			
Systems in order of per cent of mistakes	Per cent incorrect	No. of steps incorrect	No. of steps attempted
XXII	14.5	196	634
XX	13.5	139	595
I	13.4	115	861
XVII	10.5	96	918
XVIII	10.4	102	982
VI	10.1	91	908
X	9.9	81	819
IX	9.6	86	898
V	9.2	89	967
XIII	8.9	75	847
XV	8.8	72	818
VII	8.76	87	993
IV	8.5	81	951
XXV	8.3	61	739
XXIII	8	56	703
VIII	7.5	60	803
XVI	7.04	67	952
II	7	60	857
III	6.5	55	843
XII	6.3	56	896
XIV	6.2	57	921
XXIV	5.9	54	918
XXVI	5.8	54	930
XIX	5.78	57	987
XXI	5.1	44	860
XI	4.7	42	888

The Work of Courtis

Mr. S. A. Courtis,[1] then head of the department of mathematics in the Detroit Home and Day School, through the inspiration received from the work of Stone, began his investigations in 1909. From the first, his purpose was to develop a series of standard tests to be used in measuring the results of instruction in arithmetic, and he is the first to take up this particular problem. The first of the "Courtis tests" consists of eight tests known as "Series A." The first published standards were derived from the average results of testing a total of about 9000 pupils in all grades from the third to the eighth inclusive. His later standards for the same series are made up from the results obtained from 67,000 pupils distributed pretty generally throughout the country. The standards are not the average of conditions as they were found, but the estimated scores that should exist at the *end* of the year. "Series A," however, was full of defects. While more diagnostic than administrative, it was neither, yet it was evidently intended for the latter, for Courtis says, "The general purpose of testing work is not to measure the abilities of individuals to determine their fitness for promotion, but to reveal the efficiency of school procedure."

But Courtis was not blinded by enthusiasm to the defects of his first work. In 1913 he published a new set of tests and standards known as "Series B" which greatly

[1] *Courtis Standard Research Tests*, published by Department of Coöperative Research, 82 Eliot Street, Detroit, Michigan.

eliminated the defects of "Series A" from the standpoint of its purpose as expressed in the above quotation.

This new series consists of but four tests, each consisting of but a single written process, and the scores are median [1] scores rather than average scores. With the exception that Courtis has made but one test in each process, to be used in all grades from the third to the eighth inclusive, instead of a series for primary grades and another for grammar grades with a known relation between them, he has a series exceptionally well suited to measure the abilities in the four fundamental processes. Test one is an eight-minute test in written addition and consists of twenty-four exercises of equal weight, each being made of nine three-figured numbers. Such a test allows enough time devoted to a single process to eliminate more nearly the accidental errors and thus it becomes a much more reliable measure than does a test of mixed exercises given for a shorter period.

Test two is a four-minute test in written subtraction consisting of twenty-four exercises with eight-figured numbers. Test three is a six-minute test in multiplication, and test four an eight-minute test in division.

There are four forms of the same weighting in each test. One form of each is given here.

[1] The median score is found by arranging the scores in order of magnitude from the best down to the poorest and taking the middle case. Thus of 35 scores so arranged the 18th is the *median*, for there are 17 poorer and 17 better. It differs slightly from the average, for each score influences the median only as a single case, whereas, in calculating an average, very small or very large scores have a very marked effect upon the general average.

COURTIS STANDARD RESEARCH TESTS

Arithmetic Test No. 1 Addition

Series B Form 1

SCORE
No. Attempted____
No. Right________

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

927	297	136	486	384	176	277	837
379	925	340	765	477	783	445	882
756	473	988	524	881	697	682	959
837	983	386	140	266	200	594	603
924	315	353	812	679	366	481	118
110	661	904	466	241	851	778	781
854	794	547	355	796	535	849	756
965	177	192	834	850	323	157	222
344	124	439	567	733	229	953	525

537	664	634	572	226	351	428	862
695	278	168	253	880	788	975	159
471	345	717	948	663	705	450	383
913	921	142	529	819	174	194	451
564	787	449	936	779	426	666	938
932	646	453	223	123	649	742	433
559	433	924	358	338	755	295	599
106	464	659	676	996	140	187	172
228	449	432	122	303	246	281	152

677	223	186	275	432	634	547	588
464	878	478	521	876	327	197	256
234	682	927	854	571	327	685	719
718	399	516	939	917	394	678	524
838	904	923	582	749	807	456	969
293	353	553	566	495	169	393	761
423	419	216	936	250	491	525	113
955	756	669	472	833	885	240	449
519	314	409	264	318	403	152	122

COURTIS STANDARD RESEARCH TESTS

Arithmetic Test No. 2 Subtraction

Series B Form 1

SCORE
No. Attempted ____
No. Right ________

You will be given four minutes to find the answers to as many of these subtraction examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

107795491 77197029	75088824 57406394	91500053 19901563	87939983 72207316
160620971 80361837	51274387 25842708	117359208 36955523	47222970 17504943
115364741 80195261	67298125 29346861	92057352 42689037	113380936 42556840
64547329 48813139	121961783 90492726	109514632 81268615	125778972 30393060
92971900 62207032	104339409 74835938	60472960 50196521	119811864 34379846
137769153 70176835	144694835 74199225	123822790 40568814	80836465 49178036

Name.., ..., Age last birthday..............
BOY OR GIRL
School..Grade..........................Room.... ....
City...............................State...Date....................

COURTIS STANDARD RESEARCH TESTS

Arithmetic Test No. 3 Multiplication

Series B Form 1

SCORE
No. Attempted_____
No. Right________

You will be given six minutes to work as many of these multiplication examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

8246 29	3597 73	5739 85	2648 46	9537 92
4268 37	7593 640	6428 58	8563 207	2947 63
5368 95	4792 84	7942 72	3586 36	9742 59
6385 48	8736 502	5942 39	6837 680	4952 47
3876 93	9245 86	7368 74	2594 25	6495 19

Name................................, ..., Age last birthday............
BOY OR GIRL
School................................Grade..........................Room.... ¦ ....
City..............................State..............................Date..............................

COURTIS STANDARD RESEARCH TESTS

Arithmetic. Test No. 4. Division

Series B Form 1

SCORE
No. Attempted ____
No. Right ________

You will be given eight minutes to work as many of these division examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

25)6775	94)85352	37)9990	86)80066
73)58765	49)31409	68)43520	52)44252
37)14467	86)60372	94)67774	25)9750
68)39508	49)28420	52)21112	73)33653
28)23548	54)48708	39)32760	67)61707
45)33795	76)57000	93)28458	82)29602

Name................................, ... Age last birthday............
BOY OR GIRL
School..............................Grade......................Room.... ....
City............................State..............................Date..............................

Standard Scores[1]

In the table below will be found median speeds and accuracies based upon distribution of many thousands of individual scores in tests given in May or June, 1915–1916. The distribution for each grade was made up of approximately equal numbers of classes from large-city schools and from small-city and county schools. One or two of the medians have been adjusted slightly that the results as a whole might yield smooth curves. Half year divisions have been combined to make whole grades. Comparison of these results with those in the tables that follow will show the relation of the median scores, and of the values adopted as standards, to the scores from tests in various cities.

TABLE

	M.	S.	M.	S.	M.	S.	M.	S.
3	6.3	4	41	100	5.6	5	49	100
4	7.4	6	64	100	7.4	7	80	100
5	8.6	8	70	100	9.0	9	83	100
6	9.8	10	73	100	10.3	11	85	100
7	10.9	11	75	100	11.6	12	86	100
8	11.6	12	76	100	12.9	13	87	100

	M.	S.	M.	S.	M.	S.	M.	S.
3	.8	0	----	------	.6	0	----	------
4	6.2	6	67	100	4.6	4	57	100
5	7.5	8	75	100	6.1	6	77	100
6	9.1	9	78	100	8.2	8	87	100
7	10.2	10	80	100	9.6	10	90	100
8	11.5	11	81	100	10.7	11	91	100

M = Adjusted Median.
S = Scores adopted as standard.

[1] From Bulletin Number Four, *Courtis Standard Research Tests.*

Further Comments upon Series B

The attempt is to give exercises of the same weight. An examination of the exercises would seem to indicate that this has been well done except in the case of division. A glance at the answers and divisors shows a much greater variation in weight than do the other three tests. For in division the difficulty does not depend upon the number of figures involved, but upon the character of the divisors and quotients, since estimating the quotient figures constitutes the greatest difficulty.

The results show that a 7th grade pupil can try about 11 or 12 exercises in both addition and subtraction, and about 9 or 10 in both multiplication and division. This eliminates more of the chance errors of the individual and gives a more reliable result than those obtained in Series A. For this reason they become not only a measure of the efficiency of an entire school, for which they were intended, but a rather reliable measure of the abilities of the individual for promotional purposes. Yet a single attempt will not furnish a very safe measure of the individual, owing to the chance of accidental errors.

Further Comments on Series A

When Series A first appeared, the novelty of measuring any type of school efficiency by a fixed standard awakened great enthusiasm among school men throughout the country and no doubt led to a great many conclusions not warranted by the tests, and to undue emphasis upon drills to develop special abilities rather than upon the

completed process. Not only that, but in many cases no doubt undue emphasis was placed upon drill in computation to the neglect of the applicative side of the subject to the solution of problems. It is well, then, to examine the first series, which are too well known to be repeated here, to see what each of them really tests in order that teachers may examine more critically other tests that appear.

The first four tests are "speed tests" in recording the primary number combinations of addition, subtraction, multiplication, and division. It is a question whether these tests test the knowledge of the fundamental number facts as much as they do the motor dexterity of the pupil in recording the fact. That is, a high score need not necessarily show superior control of the facts, nor a low score the lack of such a control, but the score may be conditioned by the pupil's ability to make figures. But even if they do measure the control of the primary number facts, they are diagnostic tests needed in the analysis of the cause of backwardness, and not administrative tests to determine the efficiency of school instruction. The administration is chiefly interested in the finished process.

The tests, however, are not suited for diagnostic tests — a thing for which they were not intended — for the standards or scores are the work of but $1\frac{1}{2}$ minutes upon each test. Manifestly the work of an individual for $1\frac{1}{2}$ minutes cannot be taken as a reliable estimate of his ability. For all who have worked with children know how largely the element of chance enters into the work of so brief a period.

The first four tests, then, as they now stand, are not tests for measuring the abilities of the individual either to determine his fitness for promotion or to discover weakness that must be corrected in order to secure efficient work in the written processes, and they should not be used for such purposes by the teacher or supervisor.

Courtis in Bulletin Number Four says: "In June, 1912, a set of tests of the four fundamental processes in arithmetic was issued, because investigations then completed had proved that the tests of Series A were of little practical value except as instruments of research."

Test five is a speed test in copying figures and thus tests but motor control. It is thus more elementary than any of the first four and belongs even more nearly to the diagnostic class of tests than they do. As an administrative test it is of but little use, but it is a very necessary diagnostic test for determining the cause of slow work in any written process.

Test number six is called "a speed test in reasoning." In attempting to cover so wide a range as from the third to the eighth grades inclusive, the results cannot be of great value. One's ability to discover what to do in a problem depends upon his ability to read understandingly, upon his experiences through which he can interpret the situation described, and, finally, upon his knowledge of the meaning of the processes. So a set of problems that are difficult enough to be a real test of ability for eighth grade pupils would be too hard to be a test for third or fourth graders. Likewise, a test fit for children in the lower grades is too easy to test the efficiency of upper grades.

Test number seven is, by its nature, the most satisfactory test of the series from an administrative standpoint. The test is a general one covering the written processes in addition, subtraction, multiplication, and division. But the examples are not of the same weight. In order to give work for all grades from the third to the eighth inclusive, the first examples are very easy. Since the scores of the early grades are based upon the very easy examples, and since all are reckoned as equal units in making up the scores, the growth from grade to grade is much greater than appears from the scores. Hence it is not a test to be used in estimating growth from grade to grade.

Test number eight is a test in reasoning and computation combined. It consists of two-step problems. As to length of statement, all are mechanically of the same length. That is, there are about the same number of printers' *ems* in the composition of each of them. But that does not insure that they are all of equal units as to difficulty. The degree of difficulty depends upon the experiences of the pupil. But granting that the problems are equal in weight, manifestly, a test in reasoning sufficiently difficult to measure such ability of a pupil in the eighth grade is too difficult to measure the power of a third or fourth grade pupil.

The Tests a Real Contribution to Education

While the work of Courtis, being the pioneer work in this new field of investigation, has its defects, and some of his early tests were not well chosen for the purpose for which they were intended, he has rendered the cause of

education a distinct service in making the beginning. His standards, however, are based upon the present attainments of the schools examined. They should not be taken by school men as an ideal of attainment toward which their schools should work. They may be too high or too low. But surely a standard made as these were is much more probably a safe one than one made from the snap judgment of some individual, and such a standard goal of attainment is going to lead to more uniform work throughout the country.

But before we can have much more than a mere opinion of what a reasonable standard should be, there must be a much wider range of investigation. Standards based upon conditions as they are in no way show what degree of efficiency is desired; that is, they do not show a most economical use of a pupil's time. If we can find by tests how much skill attained in any grade is carried over into future school work or out into the various adult activities, and the relative loss of high or low attainments during long or short periods of disuse, and many other such problems, we shall then be in a much better position to state a more scientific standard of efficiency. That is, we will be in a much better position to decide when training in computation, or any other ability, in any grade, ceases to be an economical use of a pupil's time.

There should be established, carefully and scientifically, a measure known to all teachers, and one that can be used by them at any time and without extra expense or extra use of time. Thus, we should be able to say that ability to add twelve exercises, each consisting of six

three-figured numbers, in ten minutes, ten of the twelve sums being correct, is ability of a certain grade. Such a standard for each grade and each process would encourage a teacher to work with a definite aim in mind.

The Work of Woody

In 1916 Dr. Clifford Woody[1] made an arithmetic scale of measurement differing from those of Courtis in several particulars. Instead of all exercises being of equal weight, they ranged from the simplest primary number facts through fractions, decimals, and compound numbers. Thus they measure abilities too primary to be measured by the Courtis tests and also measure a wider range of abilities. There are two distinct series, known as "Series A" and "Series B." Twenty minutes is allowed for each scale in Series A, and ten minutes is allowed for each in Series B. The author recommends that in Series B all tests be taken in succession.

The tentative standards set up by the author were based upon a total of about 20,000 test papers. There are two methods of scoring used. The methods and scores are not reproduced here, for before the standards of achievement would be of value to a teacher she would have to know the conditions under which the tests were given and the methods of computing the scores. For these the reader is referred to Dr. Woody's monograph on the subject. To show the nature of the tests, however, Series A is reproduced here.

[1] *Measurements of Some Achievements in Arithmetic*, Teachers College, Columbia University, New York City, 1916.

The Woody Arithmetic Scale

Series A

ADDITION SCALE

Name..

When is your next birthday?..................................How old will you be?..................

Are you a boy or girl?.....................................In what grade are you?.........................

(1)
2
3

(2)
2
4
3

(3)
17
2

(4)
53
45

(5)
72
26

(6)
60
37

(7)
$3+1=$

(8)
$2+5+1=$

(9)
20
10
2
30
25

(10)
21
33
35

(11)
32
59
17

(12)
43
1
2
13

(13)
23
25
16

(14)
$25+42=$

(15)
100
33
45
201
46

(16)
9
24
12
15
19

(17)
199
194
295
156

(18)
2563
1387
4954
2065

(19)
$ .75
1.25
.49

(20)
$12.50
16.75
15.75

(21)
$8.00
5.75
2.33
4.16
.94
6.32

(22)
547
197
685
678
456
393
152
240
152

(23)
$\frac{1}{3}+\frac{1}{3}=$

(24)
4.0125
1.5907
4.10
8.673

(25)
$\frac{3}{8}+\frac{5}{8}+\frac{7}{8}+\frac{1}{8}=$

(26)
$12\frac{1}{3}$
$62\frac{1}{2}$
$12\frac{1}{3}$
$37\frac{1}{2}$

(27)
$\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=$

(28)
$\frac{2}{4}+\frac{1}{4}=$

(29)
$4\frac{3}{4}$
$2\frac{1}{4}$
$5\frac{1}{4}$

(30)
$2\frac{1}{2}$
$6\frac{3}{8}$
$3\frac{3}{4}$

(31)
113.46
49.6097
19.9
9.87
.0086
18.253
6.04

(32)
$\frac{3}{4}+\frac{1}{3}+\frac{1}{4}=$

(33)
.49
.28
.63
.95
1.69
.22
.33
.36
1.01
.56
.88
.75
.56
1.10
.18
.56

(34)
$\frac{1}{6}+\frac{3}{8}=$

(35)
2 ft. 6 in.
3 ft. 5 in.
4 ft. 9 in.

(36)
2 yr. 5 mo.
3 yr. 6 mo.
4 yr. 9 mo.
5 yr. 2 mo.
6 yr. 7 mo.

(37)
$16\frac{1}{3}$
$12\frac{1}{8}$
$21\frac{1}{2}$
$32\frac{3}{4}$

(38)
$25.091+100.4+25+98.28+19.3614=$

SERIES A

SUBTRACTION SCALE

Name..
When is your next birthday?..............................How old will you be?....................
Are you a boy or girl?........................In what grade are you?....................................

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
8 5	6 0	2 1	9 3	4 4	11 7	13 8	59 12	78 37	$7-4=$	76 60

(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
27 3	16 9	50 25	21 9	270 190	393 178	1000 537	567482 106493	$2\frac{3}{4}-1=$

(21)	(22)	(23)	(24)	(25)	(26)
10.00 3.49	$3\frac{1}{2}-\frac{1}{2}=$	80836465 49178036	$8\frac{7}{8}$ $5\frac{3}{4}$	27 $12\frac{5}{8}$	4 yd. 1 ft. 6 in. 2 yd. 2 ft. 3 in.

(27)	(28)	(29)	(30)
5 yd. 1 ft. 4 in. 2 yd. 2 ft. 8 in.	$10-6.25=$	$75\frac{3}{4}$ $52\frac{1}{4}$	$9.8063-9.019=$

(31)	(32)	(33)	(34)	(35)
$7.3-3.00081=$	1912 6 mo. 8 da. 1910 7 mo. 15 da.	$\frac{5}{12}-\frac{2}{10}=$	$6\frac{1}{8}$ $2\frac{7}{8}$	$3\frac{7}{8}-1\frac{5}{8}=$

SERIES A

MULTIPLICATION SCALE

Name..
When is your next birthday?..............................How old will you be?....................
Are you a boy or girl?........................In what grade are you?....................................

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$3\times7=$	$5\times1=$	$2\times3=$	$4\times8=$	23 3	310 4	$7\times9=$

(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
50 3	254 6	623 7	1036 8	5096 6	8754 8	165 40	235 23

(16)	(17)	(18)	(19)	(20)	(21)	(22)
7898 9	145 206	24 234	9.6 4	287 .05	24 $2\frac{1}{2}$	$8\times5\frac{3}{4}=$

(23)	(24)	(25)	(26)	(27)	(28)	(29)
$1\frac{1}{4}\times8=$	16 $2\frac{5}{8}$	$\frac{7}{8}\times\frac{3}{4}=$	9742 59	6.25 3.2	.0123 9.8	$\frac{1}{8}\times2=$

(30)	(31)	(32)	(33)	(34)
2.49 36	$\frac{12}{25}\times\frac{12}{35}=$	6 dollars 49 cents 8	$2\frac{1}{2}\times3\frac{1}{2}=$	$\frac{1}{2}+\frac{1}{2}=$

(35)	(36)	(37)	(38)	(39)
$987\frac{3}{4}$ 25	3 ft. 5 in. 5	$2\frac{1}{4}\times4\frac{1}{2}\times1\frac{1}{2}=$	$.0963\frac{1}{8}$ .084	8 ft. $9\frac{1}{2}$ in. 9

Series A

DIVISION SCALE

Name..

When is your next birthday?..............................How old will you be?....................

Are you a boy or girl?.......................In what grade are you?...................................

(1) 3)6	(2) 9)27	(3) 4)28	(4) 1)5	(5) 9)36	(6) 3)39
(7) 4 ÷ 2 =	(8) 9)0	(9) 1)1	(10) 6 × .... = 30	(11) 2)13	(12) 2 ÷ 2 =
(13) 4)24 lb. 8 oz.	(14) 8)5856	(15) $\frac{1}{4}$ of 128 =	(16) 68)2108	(17) 50 ÷ 7 =	
(18) 13)65065	(19) 248 ÷ 7 =	(20) 2.1)25.2	(21) 25)9750	(22) 2)13.50	
(23) 23)469	(24) 75)2250300	(25) 2400)504000	(26) 12)2.76		
(27) $\frac{7}{8}$ of 624 =	(28) .003).0936	(29) $3\frac{1}{2} \div 9 =$	(30) $\frac{3}{4} \div 5 =$		
(31) $\frac{5}{6} \div \frac{3}{5} =$	(32) $9\frac{5}{8} \div 3\frac{3}{4} =$	(33) 52)3756			
(34) $62.50 \div 1\frac{1}{4} =$	(35) 531)37722	(36) 9)69 lb. 9 oz.			

The Cleveland Survey Tests

A series of fifteen tests used in the Cleveland, Ohio, Survey[1] of 1915 is given here. As will be seen, they range from the primary number facts to simple operations with fractions. It was the intention of the authors of the tests to make tests that would show both the complexity of the processes which a given grade can master and also the number of examples of a given type that can be performed in a given time.

[1] *Cleveland Education Survey*, by C. H. Judd. Published by The Survey Committee of the Cleveland Foundation, Cleveland, Ohio, 1916.

The Cleveland Survey

Arithmetic Tests

June 1915

INDIVIDUAL SCORE SHEET

Name..Age ....
Grade...Date ....
School..Teacher ....

SCORE IN NUMBER OF EXAMPLES RIGHT

Set A	Set B	Set C	Set D	Set E
Set F	Set G	Set H	Set I	Set J
Set K	Set L	Set M	Set N	Set O

Total Score in Points for Whole Test..

INSTRUCTIONS FOR CHILDREN

1. Obey promptly all signals from the examiner, who will tell you when to begin working and when to stop.
2. Do all your work directly on this paper. Work steadily and rapidly, but do not hurry. Only the answers that are right will be counted.

SET A — Addition —

1	6	9	0	4	1	7	9	3	2	1	3	6
2	6	5	1	2	3	7	6	0	4	5	8	9
0	3	8	9	7	8	2	1	4	8	0	2	3
7	2	1	9	6	0	5	6	7	9	5	7	1
4	7	0	3	1	2	5	6	7	5	8	6	9
6	9	8	5	4	9	8	0	2	1	3	5	0
4	2	9	7	4	5	7	4	8	0	3	9	2
3	2	3	8	0	2	1	9	6	0	4	1	8
5	0	6	2	4	5	1	6	3	7	9	0	4
7	4	3	1	8	9	0	2	3	4	8	6	5

	Number	Value	Score
Total Number Right		1	

SET B — Subtraction —

9	7	11	8	12	1	9	13	4	12
9	3	6	1	3	0	7	8	3	6
8	11	12	5	10	6	11	15	10	12
0	9	7	1	2	0	7	8	9	4
2	7	13	3	10	1	6	15	4	8
1	5	7	2	5	1	3	9	2	3
4	10	13	10	9	5	8	17	6	11
4	7	5	1	4	5	6	9	4	8
5	12	15	5	16	7	8	16	9	11
0	9	6	3	8	0	5	7	1	4

	Number	Value	Score
Total Number Right		1	

SET C — Multiplication —

3	4	9	0	5	4	2	7	4	9
2	7	8	2	6	1	9	6	0	5
9	5	4	7	6	2	3	9	0	7
1	2	8	0	5	1	3	6	5	4
1	2	7	0	8	7	3	9	2	4
6	8	7	6	3	1	8	9	0	3
1	4	8	0	4	1	6	8	0	9
5	4	9	3	5	4	2	8	7	3
1	3	6	0	3	2	6	7	5	4
7	4	8	0	9	2	3	9	5	6

	Number	Value	Score
Total Number Right		1	

SET D — Division —

3)9	4)32	6)36	2)0	7)28	9)9	3)21
6)48	1)1	5)10	2)6	4)24	7)63	6)0
8)32	1)8	5)30	8)72	1)0	9)36	1)7
2)10	7)42	1)1	6)18	3)6	4)20	7)49
1)3	2)8	6)6	3)27	8)64	1)2	4)16
5)0	3)24	9)36	2)4	8)24	7)7	2)18
6)42	3)0	7)21	4)4	3)15	9)81	7)0

	Number	Value	Score
Total Number Right		1	

SET E — Addition —

5	2	9	2	6	1	4	9
2	8	8	8	3	4	6	7
2	8	0	5	4	2	5	1
0	5	7	0	8	5	3	5
4	1	6	6	8	4	4	3

6	2	6	8	5	4	1	3
7	7	2	5	9	0	4	7
8	3	3	1	6	8	1	2
5	4	9	3	3	5	8	9
5	1	3	8	8	5	4	6

	Number	Value	Score
Total Number Right		5	

SET F — Subtraction —

616	1248	1365	1092	716
456	709	618	472	344
1267	1335	707	816	1157
509	419	277	335	908
1355	908	519	1236	1344
616	258	324	908	818
1009	768	1269	615	854
269	295	772	527	286

	Number	Value	Score
Total Number Right		8	

SET G — Multiplication —

2345	9735	8642	6789	2345
2	5	9	2	6
9735	2468	6789	3579	2468
9	3	6	3	7
5432	9876	8642	3579	9876
4	8	5	7	4
5432	3689	2457	9863	7542
8	5	6	4	7

	Number	Value	Score
Total Number Right		7	

SET H — Fractions —

				Number	Value	Score
$\frac{3}{5}+\frac{1}{5}=$	$\frac{6}{9}-\frac{4}{9}=$	$\frac{4}{9}+\frac{1}{9}=$	$\frac{8}{9}-\frac{7}{9}=$			
$\frac{1}{9}+\frac{5}{9}=$	$\frac{3}{7}-\frac{1}{7}=$	$\frac{1}{7}+\frac{4}{7}=$	$\frac{6}{7}-\frac{2}{7}=$			
$\frac{2}{9}+\frac{4}{9}=$	$\frac{4}{5}-\frac{1}{5}=$	$\frac{5}{8}+\frac{1}{8}=$	$\frac{6}{9}-\frac{5}{9}=$			
$\frac{7}{9}+\frac{1}{9}=$	$\frac{5}{7}-\frac{2}{7}=$	$\frac{5}{9}+\frac{2}{9}=$	$\frac{8}{9}-\frac{1}{9}=$			
$\frac{1}{8}+\frac{3}{8}=$	$\frac{6}{8}-\frac{1}{8}=$	$\frac{2}{7}+\frac{1}{7}=$	$\frac{5}{9}-\frac{4}{9}=$			
$\frac{2}{9}+\frac{6}{9}=$	$\frac{4}{8}-\frac{3}{8}=$	$\frac{4}{7}+\frac{2}{7}=$	$\frac{7}{9}-\frac{5}{9}=$			
			Total Number Right		3	

SET I — Division —

				Number	Value	Score
$4\overline{)55424}$	$7\overline{)65982}$	$2\overline{)58748}$	$5\overline{)41780}$			
$9\overline{)98604}$	$6\overline{)57432}$	$3\overline{)82689}$	$6\overline{)83184}$			
$8\overline{)51496}$	$9\overline{)75933}$	$8\overline{)87856}$	$4\overline{)38968}$			
			Total Number Right		10	

SET J — Addition —

7	9	4	7	2	9	6	7	7	8	9	4	3	2
5	2	5	1	9	6	9	1	8	0	5	3	1	1
4	4	8	9	4	2	6	5	5	7	3	7	7	6
2	8	1	4	8	4	7	1	4	1	4	7	6	6
6	2	4	3	5	7	0	4	1	8	6	0	9	1
0	7	8	2	1	1	4	6	8	5	2	2	6	8
5	5	5	8	5	3	3	5	2	1	3	9	3	6
1	3	1	5	2	9	7	3	1	3	9	5	4	9
8	6	3	2	4	2	1	3	3	7	2	6	5	7
3	1	9	7	3	3	6	7	9	4	2	3	4	5
2	4	6	7	6	8	0	6	8	9	8	4	2	2
9	8	3	1	7	5	6	1	4	4	5	8	9	2
9	8	5	9	6	5	6	7	5	4	6	8	9	4

	Number	Value	Score
Total Number Right		16	

SET K — Division —

$21\overline{)441}$	$32\overline{)672}$	$23\overline{)483}$	$51\overline{)1173}$
$71\overline{)1562}$	$42\overline{)882}$	$32\overline{)992}$	$61\overline{)1342}$
$53\overline{)1166}$	$22\overline{)462}$	$21\overline{)1071}$	$52\overline{)1092}$
$51\overline{)1122}$	$41\overline{)861}$	$31\overline{)961}$	$41\overline{)1681}$
$61\overline{)1281}$	$22\overline{)484}$	$31\overline{)651}$	$33\overline{)693}$

	Number	Value	Score
Total Number Right		15	

SET L — Multiplication — | Number | Value | Score

8246 × 29	3597 × 73	5739 × 85	2648 × 46	9537 × 92
4268 × 37	7593 × 64	6428 × 58	8563 × 207	2947 × 63

Total Number Right | | 30 |

SET M — Addition—

7493	8937	8625	2123	5142	3691
9016	6345	4091	1679	0376	4526
6487	2783	3844	5555	4955	7479
7591	4883	8697	6331	9314	2087
6166	1341	7314	6808	5507	8165

5226	9149	6268	9397	7337	8243
2883	8467	7725	6158	2674	6429
2584	0251	8331	3732	9669	9298
0058	7535	5493	4641	5114	7404
2398	5223	3918	7919	8154	2575

Total Number Right | | 30 |

SET N — Division —

67)32763	48)28464	97)36084	59)29382
78)69888	88)34496	69)40296	38)26562

Total Number Right | | 30 |

SET O — Fractions —

				Number	Value	Score
$\frac{11}{15}+\frac{1}{6}=$	$\frac{9}{14}-\frac{1}{4}=$	$\frac{3}{4}\times\frac{5}{6}=$	$\frac{20}{21}\div\frac{1}{6}=$			
$\frac{5}{12}+\frac{2}{8}=$	$\frac{5}{6}-\frac{2}{21}=$	$\frac{5}{6}\times\frac{19}{20}=$	$\frac{11}{12}\div\frac{5}{8}=$			
$\frac{3}{4}+\frac{3}{18}=$	$\frac{3}{8}-\frac{3}{10}=$	$\frac{1}{6}\times\frac{3}{10}=$	$\frac{5}{6}\div\frac{11}{15}=$			
			Total Number Right		30	

Instructions for Examiners

Have the children fill out the blanks at the top of the first page. Have them start and stop work together. Let there be an interval of half a minute between each set of examples. The time allowances given below must be followed exactly.

Set A..........30 seconds	Set F..........1 minute	Set K......2 minutes
Set B..........30 seconds	Set G..........1 minute	Set L......3 minutes
Set C..........30 seconds	Set H........30 seconds	Set M......3 minutes
Set D..........30 seconds	Set I..........1 minute	Set N......3 minutes
Set E..........30 seconds	Set J..........2 minutes	Set O......3 minutes

Have the children exchange papers. Read the answers aloud and let the children mark each example that is correct, "C." For each set let them count the number of C's and write the number at the end of the set in the first column; also on the first page.

The teacher should multiply the number of examples right by the number in the second column, writing the result in the third column. Then find the sum of the scores in the third column for a total score.

STANDARD SCORES

MEDIANS IN EACH ARITHMETIC TEST FOR ALL GRADES

Test	Grade					
	3	4	5	6	7	8
A	13.4	17.8	22.2	24.8	26.7	27.5
B	9.3	13.4	17.2	19.8	21.5	26.0
C	6.5	12.0	15.5	16.6	17.7	19.0
D	6.3	12.4	15.7	18.5	20.8	22.5
E	4.3	5.3	6.3	6.8	7.5	7.8
F	2.0	4.9	6.7	7.5	8.6	10.1
G	2.0	3.9	5.2	5.5	5.9	6.6
H	0.0	0.0	5.0	5.5	7.7	8.5
I	0.6	1.1	2.0	3.1	4.0	4.7
J	1.9	3.2	4.0	4.4	4.9	5.7
K	0.0	4.0	6.8	8.5	10.1	12.5
L	0.0	1.7	2.5	2.8	3.2	3.9
M	1.4	2.5	3.2	3.8	4.4	5.1
N	0.0	0.8	1.3	1.7	2.0	2.6
O	0.0	0.0	0.0	3.1	4.1	5.5

Some Cautions to Observe

It must be observed by teachers and supervisors that the "standard tests" which we have discussed are but measures of the mechanical aspects of the subject of arithmetic. There is but little if any relation between the abilities to compute and the power to interpret a problem and reason out what processes to apply. There is a danger that this may be overlooked by some and that a

teacher may overemphasize the mechanical side of the subject, which is so easily measured, to the neglect of the more important part of it, for which there can be no such definite tests.

For just as abilities to spell and to write are of no value unless we have thoughts to express through such abilities, so abilities to compute are of no value unless we can interpret a problem and know what processes to apply to its solution.

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